The Core of a Set-Valued Mapping and the Finiteness Principle for Lipschitz Selections

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The Twelfth Whitney Problems Workshop

The University of Texas at Austin, TX

August 5-9, 2019

1. Lipschitz Selection Problem: Main Settings

- $(\mathcal{M}, \rho)$ - a pseudometric space.

Thus, $\rho : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_+$ is symmetric and satisfies the triangle inequality,

but $\rho(x, y)$ may admit the value 0 for $x \neq y$.

- $(Y, \|\cdot\|)$ - a Banach space.

- $B_Y(a, r)$ - a ball of radius $r > 0$ centered at a point $a \in Y$; $B_Y = B_Y(0, 1)$.

- $\mathrm{Lip}(\mathcal{M}; Y)$ - the space of Lipschitz continuous mappings $f : \mathcal{M} \to Y$, with the seminorm

$$\|f\|_{\mathrm{Lip}(\mathcal{M};Y)} := \inf\{\lambda > 0 : \|f(x) - f(y)\| \le \lambda\,\rho(x, y),\ \ x, y \in \mathcal{M}\}$$

Lipschitz Selection Problem: Main Settings

- $\mathcal{K}_m(Y)$ - the family of all nonempty convex compact subsets of Y of dimension at most m.

- $F : \mathcal{M} \to \mathcal{K}_m(Y)$ - a set-valued mapping from $\mathcal{M}$ into $\mathcal{K}_m(Y)$.

- A (single valued) mapping $f : \mathcal{M} \to Y$ is called a *selection of F* if

$$f(x) \in F(x) \quad \text{for all} \quad x \in \mathcal{M}$$

- A selection f is said to be *Lipschitz* if $f \in \mathrm{Lip}(\mathcal{M}; Y)$.

Lipschitz Selection Problem: Main Settings

- Given $A, B \subset Y$ we let $A + B$ denote the Minkowski sum of A and B

$$A + B = \{a + b : a \in A,\ b \in B\}$$

- Let $A, A' \subset Y$. We let $\mathrm{d_H}(A, A')$ denote the Hausdorff distance between these sets:

$$\mathrm{d_H}(A, A') = \inf\{r > 0 : A + B_Y(0, r) \supset A',\ A' + B_Y(0, r) \supset A\}.$$

Lipschitz Selection Problem

Let $(\mathcal{M},\rho)$ be a pseudometric space and let $F:\mathcal{M}\to\mathcal{K}_m(Y)$ be a set-valued mapping .

1. How can we decide

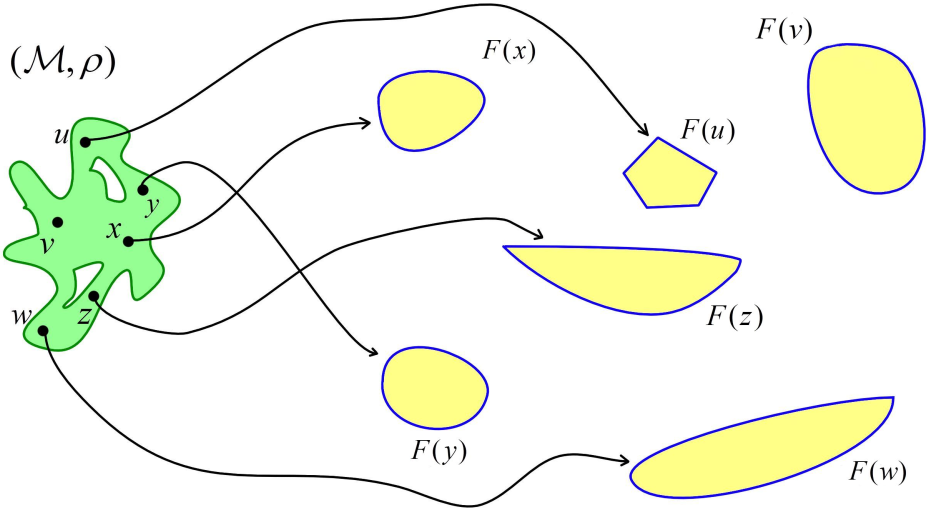
whether there exists a Lipschitz selection of F,

i.e., a mapping $f\in \mathrm{Lip}(\mathcal{M};Y)$ such that $f(x)\in F(x)$ for all $x\in\mathcal{M}$?

2. Consider the Lipschitz norms of all Lipschitz selections of F.

How small can these norms be?

This is a purely geometrical problem about a **suitable choice of points in a family convex compact sets in Y indexed by points of the metric space $\mathcal{M}$.**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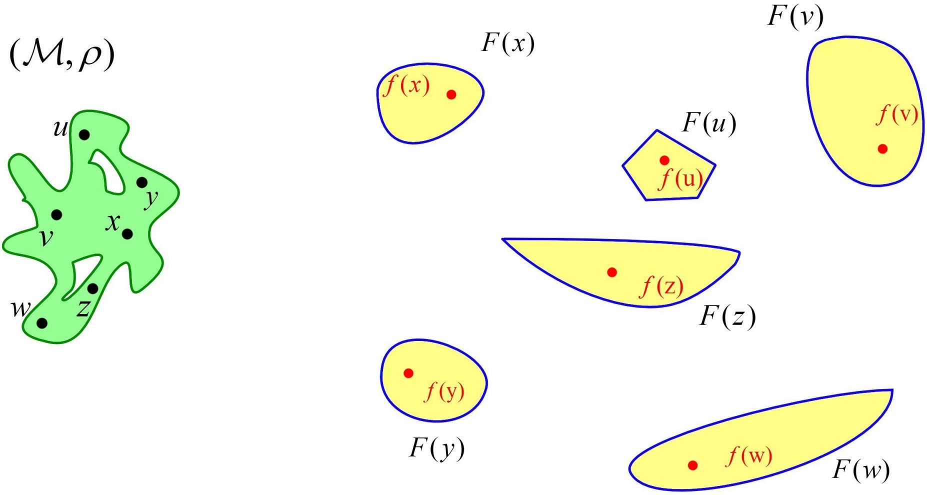
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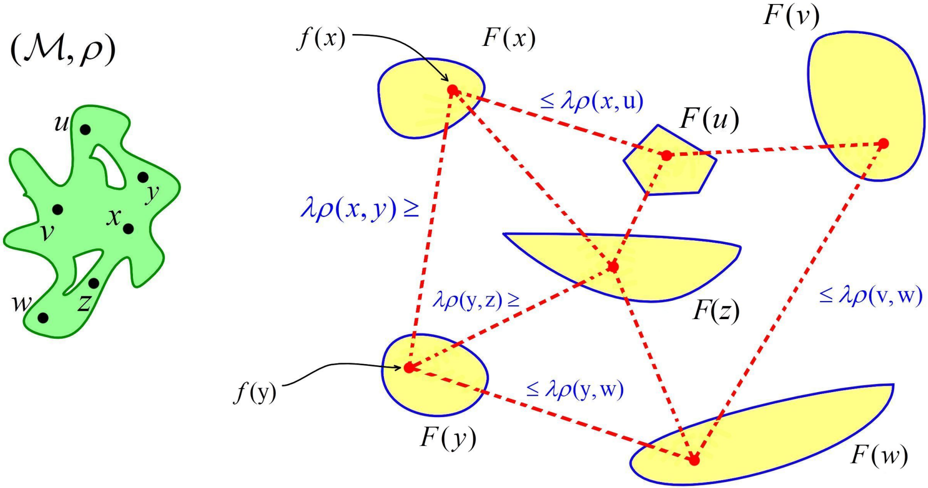

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≤ λρ(y,w)
F(y)
F(w)

Let

$$\mathbf{N(m, Y) = 2^{\min\{m+1, \dim Y\}}}$$

Theorem 1. (Fefferman, Shvartsman [2018], GAFA)

Let $(\mathcal{M}, \rho)$ be a pseudometric space and let $F : \mathcal{M} \to \mathcal{K}_m(Y)$.

Assume that for every subset $\mathcal{M}' \subset \mathcal{M}$ with $\#\mathcal{M}' \le N(m, Y)$, the restriction $F|_{\mathcal{M}'}$ of F to $\mathcal{M}'$ has a Lipschitz selection

$$f_{\mathcal{M}'} : \mathcal{M}' \to Y \quad \text{with} \quad \|f_{\mathcal{M}'}\|_{\mathrm{Lip}(\mathcal{M}', Y)} \le 1.$$

Then F has a Lipschitz selection

$$f : \mathcal{M} \to Y \quad \text{with} \quad \|f\|_{\mathrm{Lip}(\mathcal{M}, Y)} \le \gamma(m).$$

Helly's Theorem

Let $\rho \equiv 0$ on $\mathcal{M}$. In this case the Finiteness Principle holds with

$$n(m, Y) = \min\{m + 2, \dim Y + 1\}.$$

Indeed, $f \in \mathrm{Lip}((\mathcal{M}, \rho), Y) \Longleftrightarrow f(x) = f(y), x, y \in \mathcal{M} \Longrightarrow f(x) = c$ on $\mathcal{M}$.

Therefore, F has a selection $\Longleftrightarrow \exists\, c \in F(x)$ for all $x \in \mathcal{M} \Longleftrightarrow$

The family $\{F(x) : x \in \mathcal{M}\}$ has a common point

Helly's Intersection Theorem

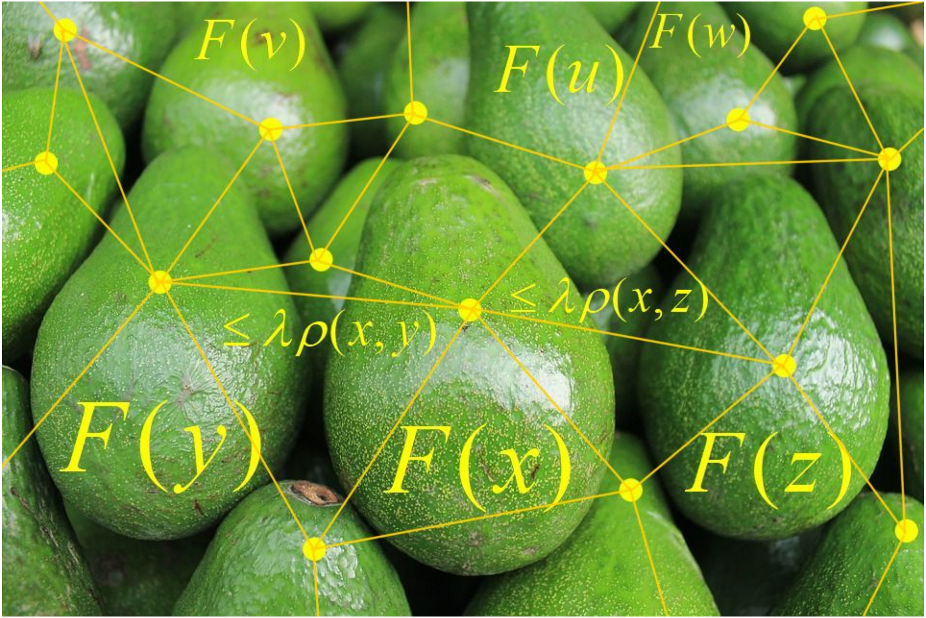
Let $\mathcal{K}$ be a family of convex compact subsets of Y of dimension at most m.

Suppose that for every subfamily $\mathcal{K}'$ of $\mathcal{K}$ consisting of at most $n(m, Y)$ elements

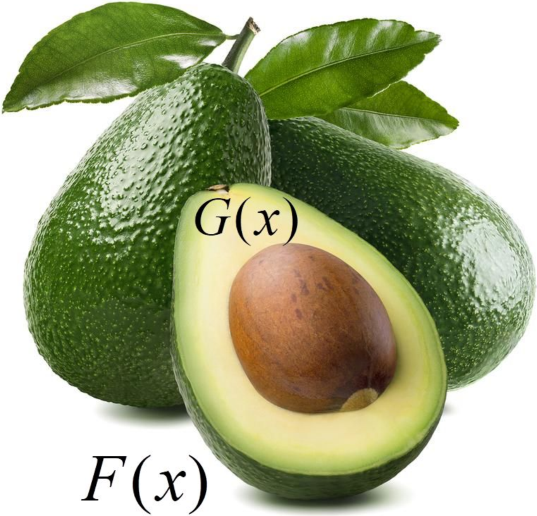
$$\bigcap_{K \in \mathcal{K}'} K \neq \emptyset.$$

Then there exists a point common to all of the family $\mathcal{K}$.

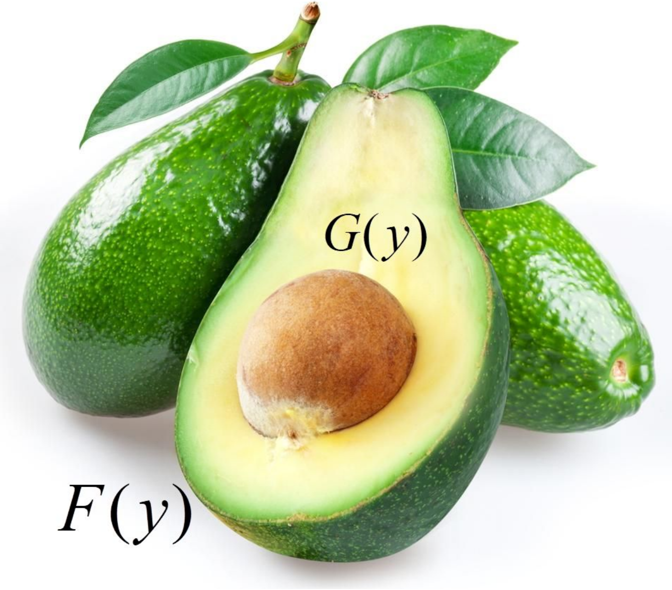
The Core of a Set-valued Mapping: an Example



The Core of a Set-valued Mapping: an Example

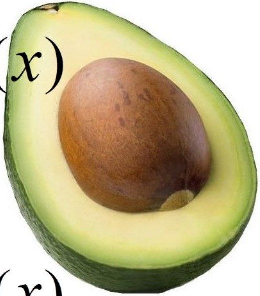


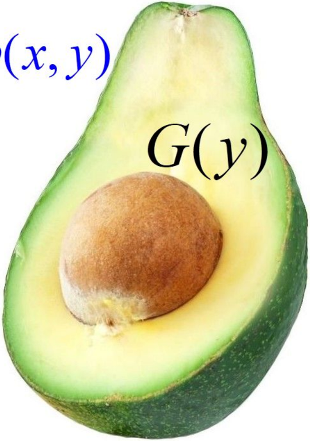
The Core of a Set-valued Mapping: an Example



The Core of a Set-valued Mapping: an Example

$$\mathrm{d}_{\mathrm{H}}(G(x),G(y)) \le \gamma\,\rho(x,y)$$





The Core of a Set-valued Mapping: Definition

Let $(\mathcal{M}, \rho)$ be a metric space and let $F : \mathcal{M} \to \mathcal{K}_m(Y)$ be a set-valued mapping. Let $\gamma > 0$.

Definition 2.

A set-valued mapping $G : \mathcal{M} \to \mathcal{K}_m(Y)$ is said to be a γ-core of the set-valued mapping F if:

(i) $G(x) \subset F(x)$ for all $x \in \mathcal{M}$.

(ii) For every $x, y \in \mathcal{M}$

$$\mathrm{d_H}(G(x), G(y)) \le \gamma\, \rho(x, y)$$

In particular, any Lipschitz selection of F with Lipschitz constant γ is a 0-dimensional γ-core of F.

Claim 3.

Let $G : \mathcal{M} \to \mathcal{K}_m(Y)$ be a γ-core of a set-valued mapping $F : \mathcal{M} \to \mathcal{K}_m(Y)$.

Then F has a Lipschitz selection $f : \mathcal{M} \to Y$ with

$$\|f\|_{\mathrm{Lip}(\mathcal{M},Y)} \le C\,\gamma$$

where $C = C(m)$ is a constant depending only on m.

The proof is immediate from the following result.

4. Steiner-type selectors

Let $\mathcal{K}(Y) = \cup\{\mathcal{K}_m(Y) : m \in \mathbb{N}\}$ be the family of all non-empty finite dimensional convex compact subsets of Y.

Theorem 4. (Sh. [2004])

There exists a mapping $S_Y : \mathcal{K}(Y) \to Y$ such that

(i). $S_Y(K) \in K$ for each $K \in \mathcal{K}(Y)$;

(ii). For every $K_1, K_2 \in \mathcal{K}(Y)$,

$$\|S_Y(K_1) - S_Y(K_2)\| \le \gamma \, \mathrm{d_H}(K_1, K_2),$$

Here $\gamma = \gamma(\dim K_1, \dim K_2)$.

We refer to $S_Y(K)$ as a *Steiner-type point* of a convex set $K \in \mathcal{K}(Y)$.

We call $S_Y : \mathcal{K}(Y) \to Y$ *a Steiner-type selector*.

Proof of Claim 3.

We define the required Lipschitz selection $f : \mathcal{M} \to Y$ of the set valued mapping $F : \mathcal{M} \to \mathcal{K}_m(Y)$ as a composition of the γ-core $G : \mathcal{M} \to \mathcal{K}_m(Y)$ and the Steiner-type selector $S_Y : \mathcal{K}(Y) \to Y$:

$$f(x) = S_Y(G(x)), \quad x \in \mathcal{M}.$$

Then,

$$f(x) = S_Y(G(x)) \in G(x) \subset F(x)$$

i.e., f is a *selection* of F.

Furthermore,

$$\begin{aligned} \|f(x) - f(y)\| &= \|S_Y(G(x)) - S_Y(G(y))\| \\ &\le C(\dim G(x), \dim G(y))\ \mathrm{d_H}(G(x), G(y)) \\ &\le C(m)\,\gamma\,\rho(x, y). \end{aligned}$$

This proves that f is a *Lipschitz selection* of F with $\|f\|_{\mathrm{Lip}(\mathcal{M},Y)} \le C(m)\,\gamma$.

The paper "Sharp Finiteness Principles for Lipschitz Selections", GAFA, 2018 by C. Fefferman and P. Shvartsman:

Given a set-valued mapping $F : \mathcal{M} \to \mathcal{K}_m(Y)$ satisfying the hypothesis of the Finiteness Principle for Lipschitz Selections (Theorem 1) we construct a γ-core with $\gamma = \gamma(m)$. We do this in three steps.

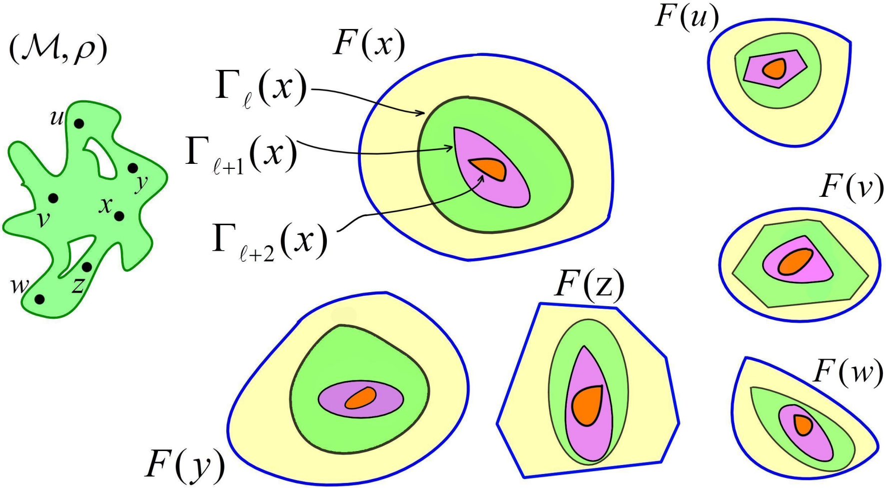
Step 1. We introduce a family $\Gamma_\ell : \mathcal{M} \to \mathcal{K}_m(Y),\ \ \ell = 0, 1, ...,$ of the so-called Basic Convex Sets having the following properties:

- (i) $\Gamma_\ell(x) \neq \emptyset$ and $\Gamma_\ell(x) \subset F(x)$ for every $x \in \mathcal{M}$, $\ell = 0, 1, ...$;
- (ii) For all $x, y \in \mathcal{M}$ and $\ell = 0, 1, ...,$

$$\Gamma_{\ell+1}(x) \subset \Gamma_\ell(y) + \ B_Y(0, \lambda\rho(x, y))$$

with some $\lambda = \lambda(m)$.

In particular, $\Gamma_{\ell+1}(x) \subset \Gamma_\ell(x)$, for all $\ell = 0, 1, ...$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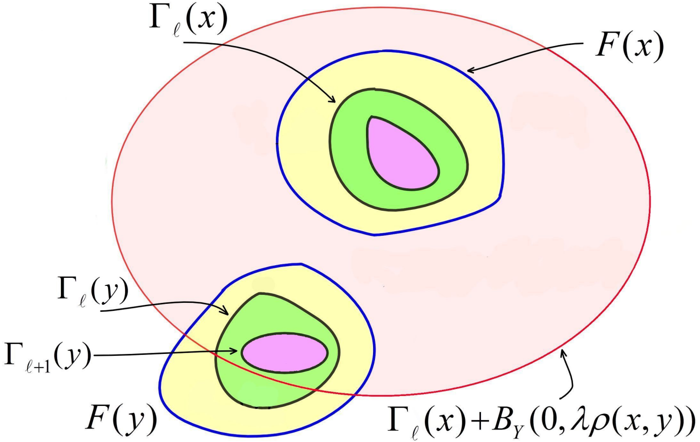


Apparently, in general, the family of mappings

$$\Gamma_\ell : \mathcal{M} \to \mathcal{K}_m(Y), \quad \ell = 0, 1, ...,$$

is not a core of the set-valued mapping F (for any $\ell = 0, 1, ...$.)

Step 2. We prove that the Finiteness Principle for Lipschitz selections holds for any finite metric tree.

The proof relies on ideas developed in the paper

C. Fefferman, A. Israel, K. Luli

"Finiteness Principles for Smooth Selection", GAFA, 2016.

for the case $\mathcal{M} = \mathbb{R}^n$.

Step 2 is the most technically difficult part of our proof.

Step 3. We construct a core of the set-valued mapping $F : \mathcal{M} \to \mathcal{K}_m(Y)$ as intersection of orbits of Lipschitz selections with respect to a certain family of metric trees with vertices in $\mathcal{M}$.

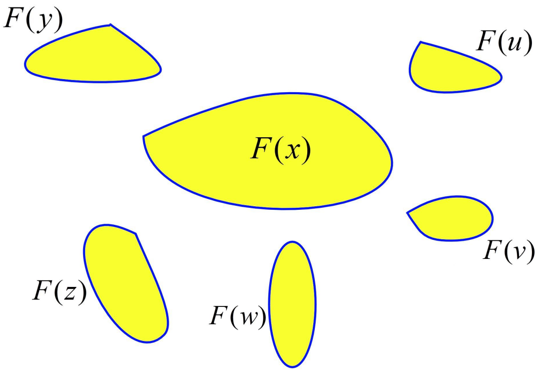
6. λ-Balanced Refinements

Let $F : \mathcal{M} \to \mathcal{K}_m(Y)$ be a set-valued mapping, an let $\lambda \ge 0$. Let

$$\mathcal{BR}[F:\lambda](x) = \bigcap_{z\in\mathcal{M}} \left[F(z) + \lambda\,\rho(x,z)\,B_Y\right], \qquad x \in \mathcal{M}.$$

We refer to the set-valued mapping $\mathcal{BR}[F:\lambda] : \mathcal{M} \to \mathcal{K}_m(Y)$ as a

λ-balanced refinement of the mapping F.



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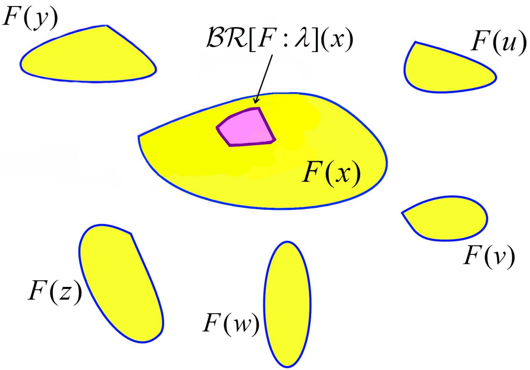


Clearly, $\mathcal{BR}[F:\lambda](x)$ is a convex compact subset of Y, and

$$\mathcal{BR}[F:\lambda](x) \subset F(x)$$

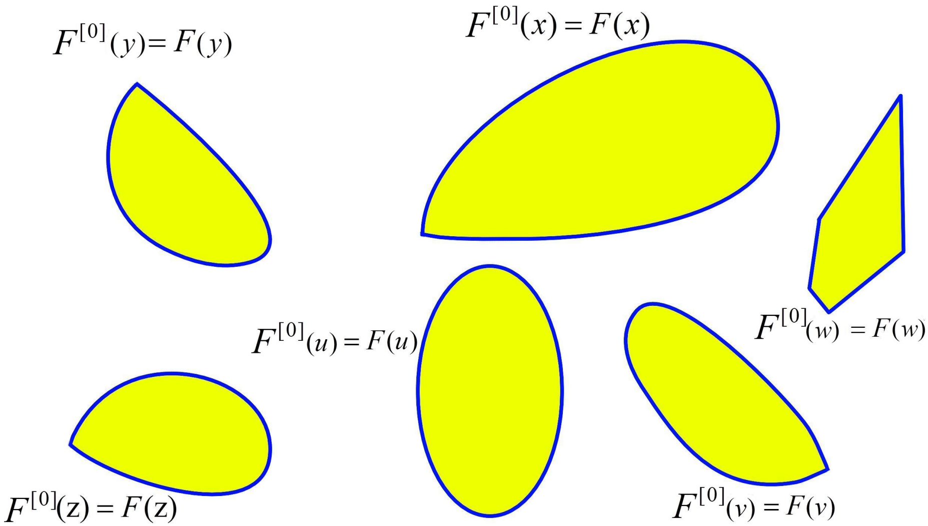
for all $x \in \mathcal{M}$.

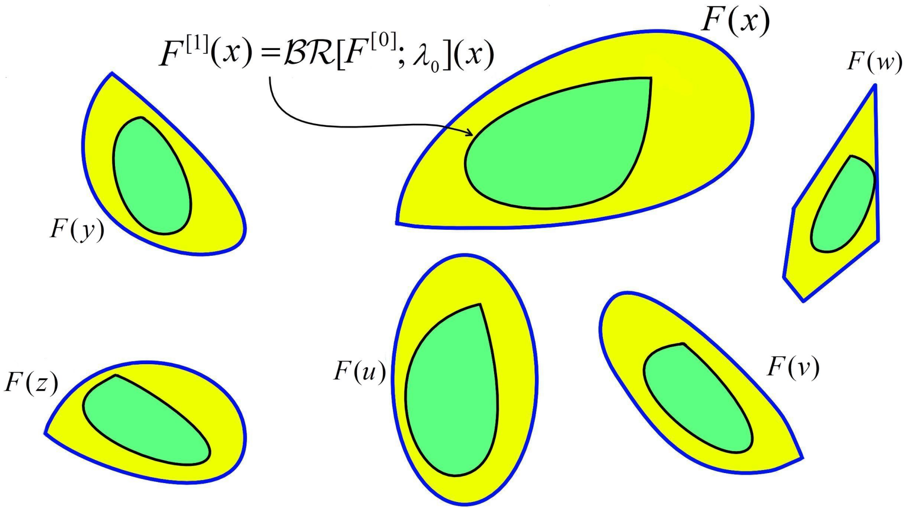
Let $\vec{\lambda} = \{\lambda_0, \lambda_1, ..., \lambda_\ell\}$ where $1 \le \lambda_k \le \lambda_{k+1}, \quad k = 1, ..., \ell - 1$.

We set $F^{[0]} = F$, and

$$F^{[k+1]}(x) = \mathcal{BR}[F^{[k]}:\lambda_k](x) = \bigcap_{z \in \mathcal{M}} \left[F^{[k]}(z) + \lambda_k \, \rho(x,z) \, B_Y\right]$$

for every $x \in \mathcal{M}$ and $k \in \mathbb{N}$.


$F^{[0]}(y) = F(y)$
$F^{[0]}(x) = F(x)$
$F^{[0]}(w) = F(w)$
$F^{[0]}(u) = F(u)$
$F^{[0]}(z) = F(z)$
$F^{[0]}(v) = F(v)$


$F^{[1]}(x) = \mathcal{BR}[F^{[0]}; \lambda_0](x)$
$F(x)$
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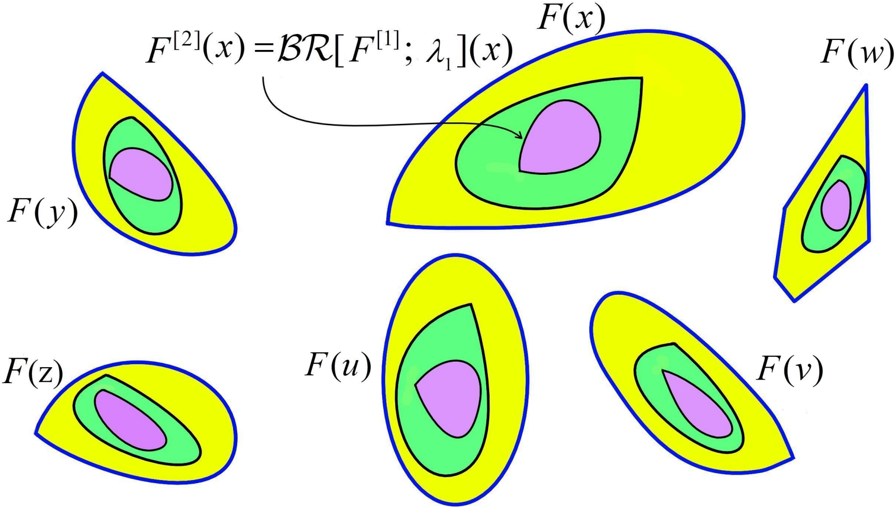
$F^{[2]}(x) = \mathcal{BR}[F^{[1]};\ \lambda_1](x)$
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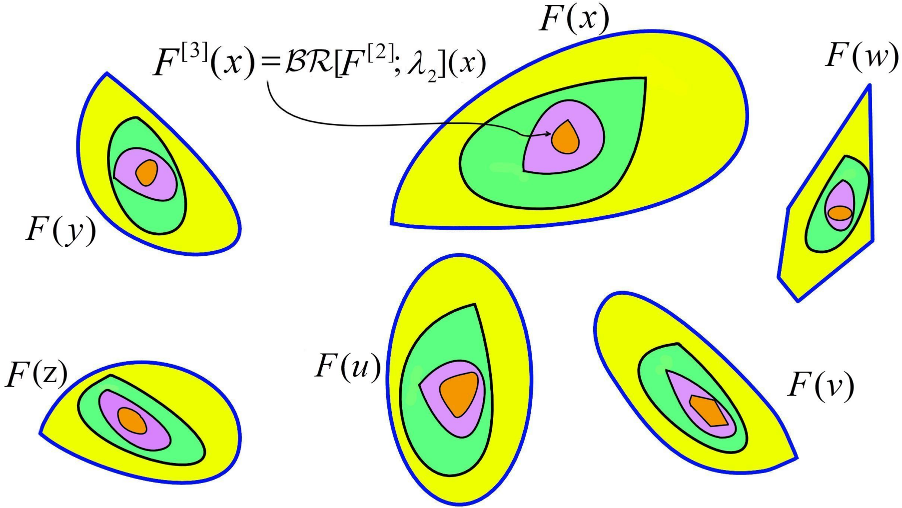
$F^{[3]}(x) = \mathcal{BR}[F^{[2]}; \lambda_2](x)$
$F(x)$
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$F(z)$
$F(u)$
$F(v)$

Conjecture 5.

Let $m \in \mathbb{N}$. There exist constants $\ell = \ell(m) \in \mathbb{N}$, $\gamma = \gamma(m) \geq 1$, and a non-decreasing positive sequence of parameters

$$\vec{\lambda} = \{\lambda_0(m), \lambda_2(m), ..., \lambda_\ell(m)\},$$

such that the following holds:

Let $F : \mathcal{M} \to \mathcal{K}_m(Y)$ be a set-valued mapping such that for every subset $\mathcal{M}' \subset \mathcal{M}$ with $\#\mathcal{M}' \leq N(m, Y)$, the restriction $F|_{\mathcal{M}'}$ of F to $\mathcal{M}'$ has a Lipschitz selection $f_{\mathcal{M}'} : \mathcal{M}' \to Y$ with $\|f_{\mathcal{M}'}\|_{\mathrm{Lip}(\mathcal{M}',Y)} \leq 1.$

Then the set-valued mapping

$$F^{[\ell]} : \mathcal{M} \to \mathcal{K}_m(Y) \quad \text{is a} \quad \gamma\text{-core} \quad \text{of} \quad F.$$

Recall that $F^{[\ell]}$ is a γ-core if

$$\mathrm{d_H}(F^{[\ell]}(x), F^{[\ell]}(y)) \le \gamma\,\rho(x,y), \quad x, y \in \mathcal{M}.$$

Thus,

$$F^{[\ell]}(x) \subset F^{[\ell]}(y) + \gamma\,\rho(x,y) B_Y, \quad x, y \in \mathcal{M}.$$

Let us reformulate this property in terms of γ-balanced refinements.

Given $x \in \mathcal{M}$ we have:

$$F^{[\ell+1]}(x) = \mathcal{BR}\,[F^{[\ell]}\!:\!\gamma](x) = \bigcap_{y \in \mathcal{M}} \left[F^{[\ell]}(y) + \gamma\,\rho(x,y)\,B_Y\right]$$

so that $F^{[\ell+1]}(x) \supset F^{[\ell]}(x)$ proving that

$$F^{[\ell+1]} = F^{[\ell]} \quad \text{on} \quad \mathcal{M}.$$

Conjecture 5.1: Stabilization Property of λ-Balanced Refinements

Given $m \in \mathbb{N}$ there exist $\ell = \ell(m) \in \mathbb{N}$ and a non-decreasing positive sequence

$$\vec{\lambda} = \{\lambda_0(m), \lambda_2(m), ..., \lambda_\ell(m)\}$$

such that for every set-valued mapping $F : \mathcal{M} \to \mathcal{K}_m(Y)$ satisfying the hypothesis of the Finiteness Principle the following Stabilization Property

$$F^{[\ell+1]}(x) = F^{[\ell]}(x) \neq \emptyset \quad \text{for all} \quad x \in \mathcal{M},$$

holds.

Theorem 6.

Let $(\mathcal{M}, \rho)$ be a pseudometric space.

Conjecture 5 holds with

$$\ell = 2 \quad \text{(two iterations)}, \quad \vec{\lambda} = \{2^6, 2^7\} \quad \text{and} \quad \gamma = 2^{14}$$

whenever:

(i) $m = 1$ and Y is an arbitrary Banach space;

(ii) $m = 2$ and $\dim Y = 2$.

Conjecture 5: $m = 2$ and $\dim Y = 2$

A Sketch of the Proof.

The finiteness constant $N(2, Y) = 4$ provided $\dim Y = 2$.

We know that for every subset $\mathcal{M}' \subset \mathcal{M}$ with $\#\mathcal{M}' \le 4$, the restriction $F|_{\mathcal{M}'}$ of F to $\mathcal{M}'$ has a Lipschitz selection

$$f_{\mathcal{M}'} : \mathcal{M}' \to Y \quad \text{with} \quad \|f_{\mathcal{M}'}\|_{\mathrm{Lip}(\mathcal{M}', Y)} \le 1.$$

Proposition 7. (Sh. [2002])

For every subset

$$S \subset \mathcal{M} \quad \text{with} \quad \#S \le 10$$

the restriction $F|_S$ of F to S has a Lipschitz selection $f_S : S \to \mathbb{R}^2$ with the Lipschitz seminorm

$$\|f_S\|_{\mathrm{Lip}(S, \mathbb{R}^2)} \le 2^6.$$

Let $B = B_Y$. We introduce a new metric on $\mathcal{M}$:

$$\mathrm{d}(x,y) = 2^6 \rho(x,y), \quad x,y \in \mathcal{M}.$$

Then the following assumption holds:

Assumption 8.

For every subset $S \subset \mathcal{M}$ with $\#S \leq 10$ the restriction $F|_S$ has a Lipschitz (with respect to d) selection $f_S : S \to \mathbb{R}^2$ with the Lipschitz seminorm

$$\|f_S\|_{\mathrm{Lip}((S,d),\mathbb{R}^2)} \leq 1.$$

We proceed two balanced refinements of F (with respect to the metric d) with the parameters $\vec{\lambda} = \{1,2\}$:

$$F^{[1]}(x) = \bigcap_{z \in \mathcal{M}} [F(z) + \mathrm{d}(x,z)\, B], \quad x \in \mathcal{M},$$

and

$$G(x) = F^{[2]}(x) = \mathcal{BR}[F^{[1]}:2] = \bigcap_{z \in \mathcal{M}} \left[F^{[1]}(z) + 2\,\mathrm{d}(x,z)\, B\right], \quad x \in \mathcal{M}.$$

Thus,

$$G(x)=\bigcap_{z\in\mathcal{M}}\left\{\left(\bigcap_{z'\in\mathcal{M}}[F(z')+\mathrm{d}(z,z')\,B]\right)+2\,\mathrm{d}(x,z)\,B\right\},\quad x\in\mathcal{M}.$$

Clearly,

$$G(x)\subset F(x),\quad x\in\mathcal{M}.$$

We prove that the set-valued mapping

$$G:\mathcal{M}\to\mathcal{K}_2(Y)\ \text{ is a }\ \gamma-\text{core of }\ F$$

(with respect to d) with $\gamma=162=2\cdot 9^2$.

Thus, our aim is prove that

(i) $G(x)\neq\emptyset$ for every $x\in\mathcal{M}$;

(ii) $\mathrm{d}_{\mathrm{H}}(G(x),G(y))\le\gamma\,\mathrm{d}(x,y)$ for all $x,y\in\mathcal{M}$.

The proof of part (i) relies on the following corollary of Helly's Theorem:

Lemma 9.

Let $\mathcal{K}$ be a collection of convex compact subsets of $\mathbb{R}^2$.

Suppose that

$$\bigcap_{K \in \mathcal{K}} K \neq \emptyset\,.$$

Then for every $r \geq 0$ the following equality

$$\left(\bigcap_{K \in \mathcal{K}} K\right) + B(0,r) = \bigcap_{K,K' \in \mathcal{K}} \left\{ \left[K \bigcap K' \right] + B(0,r) \right\}$$

holds.

We recall that

$$G(x) = \bigcap_{z\in\mathcal{M}} \left\{ \left(\bigcap_{z'\in\mathcal{M}} [F(z') + \mathrm{d}(z,z')\,B] \right) + 2\,\mathrm{d}(x,z)\,B \right\}, \quad x \in \mathcal{M}.$$

This and Lemma 9 imply the following representation of the set $G(x)$:

Lemma 10.

For every $x \in \mathcal{M}$

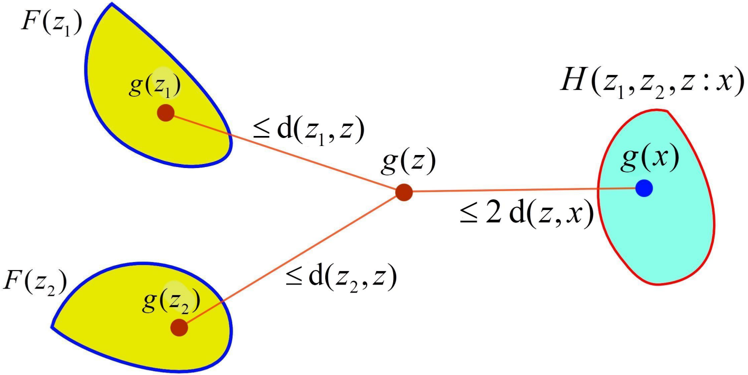
$$G(x) = \bigcap_{z,z_1,z_2\in\mathcal{M}} \left\{ \left([F(z_1)+\mathrm{d}(z_1,z)B] \bigcap [F(z_2)+\mathrm{d}(z_2,z)B] \right) + 2\mathrm{d}(z,x)B \right\}$$

Given $x, z, z_1, z_2 \in \mathcal{M}$, let

$$H(z_1, z_2, z : x) = \Big\{[F(z_1) + \mathrm{d}(z_1, z)B] \bigcap [F(z_2) + \mathrm{d}(z_2, z)B]\Big\} + 2\,\mathrm{d}(z, x)B.$$

$$a \in H(z_1, z_2, z : x) \Longleftrightarrow \exists\ \ g(z_1) \in F(z_1),\ g(z_2) \in F(z_2),\ g(z) \in \mathbb{R}^2,\ g(x) = a,$$

$$\|g(z) - g(z_1)\| \le \mathrm{d}(z, z_1), \quad \|g(z) - g(z_2)\| \le \mathrm{d}(z, z_2), \quad \|g(x) - g(z)\| \le 2\,\mathrm{d}(z, x).$$



Thus,

$$G(x) = \bigcap_{z, z_1, z_2 \in \mathcal{M}} H(z_1, z_2, z : x)$$

This representation, Helly's Theorem in $\mathbb{R}^2$ and Assumption 8 readily imply the required property (i):

$$G(x) \neq \emptyset, \quad x \in \mathcal{M}.$$

Prove property (ii) which is equivalent to the following imbeddings:

$$G(x) + \gamma \, \mathrm{d}(x, y) B \supset G(y) \quad x, y \in \mathcal{M},$$

and

$$G(y) + \gamma \, \mathrm{d}(x, y) B \supset G(x), \quad x, y \in \mathcal{M}.$$

Given $x, y \in \mathcal{M}$ let us prove that

$$G(x) + \gamma \, \mathrm{d}(x, y) B \supset G(y)$$

Lemma 9 and 10 tell us:

$$G(x) + \gamma\, \mathrm{d}(x,y)B = \left[\bigcap_{z,z_1,z_2\in\mathcal{M}} H(z_1,z_2,z:x)\right] + \gamma\, \mathrm{d}(x,y)B =$$

$$\bigcap_{\mathcal{A}\subset\mathcal{M}} \left\{\left[H(u_1,u_2,u:x)\bigcap H(v_1,v_2,v:x)\right] + \gamma\, \mathrm{d}(x,y)\, B\right\}$$

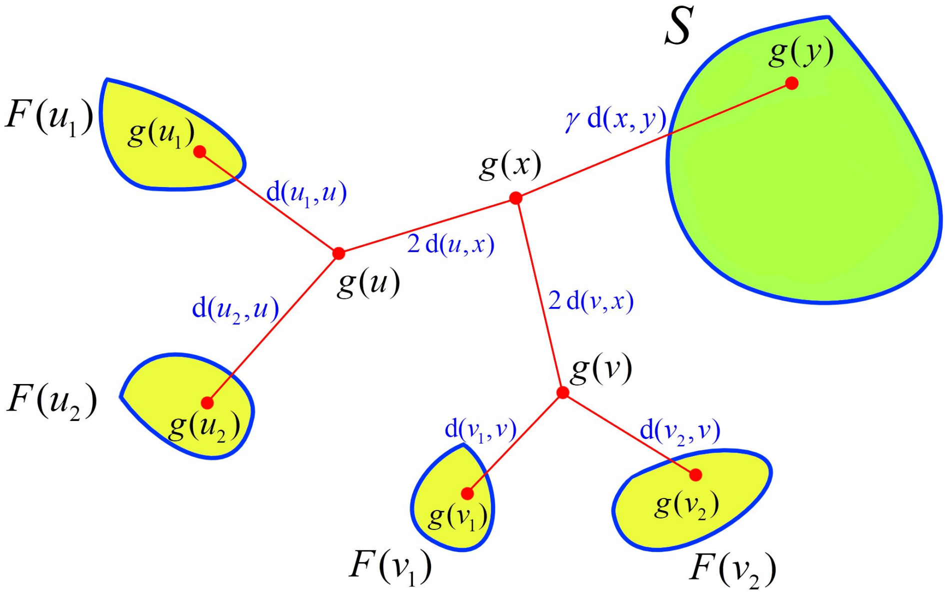
where $\mathcal{A} = \{u,u_1,u_2,v,v_1,v_2,x\}$ runs over all subsets of $\mathcal{M}$ with $\#\mathcal{A} \le 7$.

Fix $\mathcal{A} = \{u,u_1,u_2,v,v_1,v_2,x\} \subset \mathcal{M}$. Let

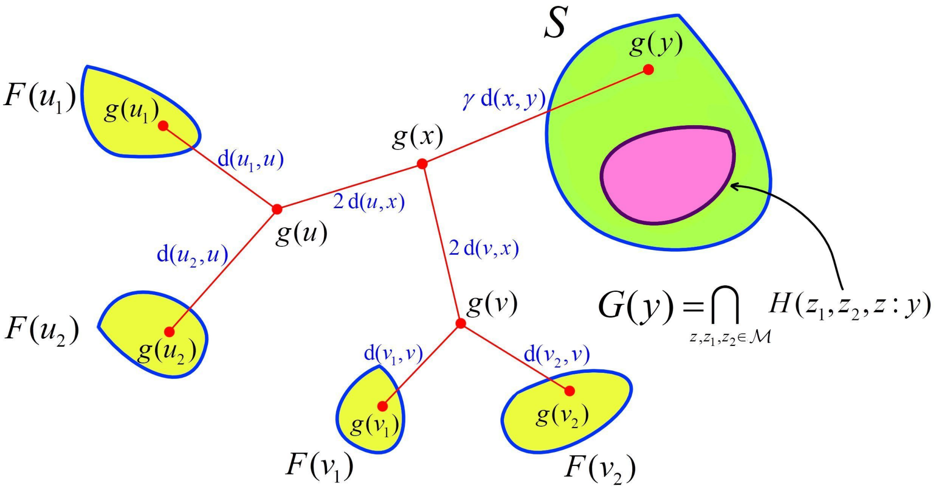
$$S = \left[H(u_1,u_2,u:x)\bigcap H(v_1,v_2,v:x)\right] + \gamma\, \mathrm{d}(x,y)\, B.$$

Prove that

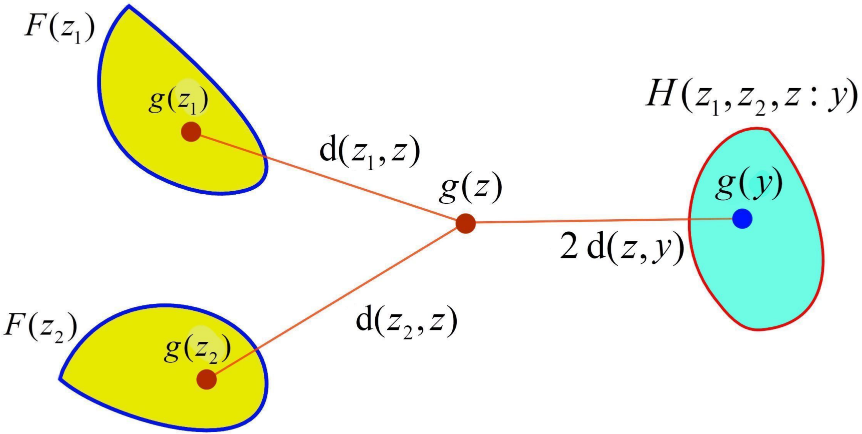
$$S \supset G(y) = \bigcap_{z,z_1,z_2\in\mathcal{M}} H(z_1,z_2,z:y).$$



S
g(y)
γ d(x, y)
F(u₁)
g(u₁)
d(u₁, u)
g(x)
2 d(u, x)
g(u)
d(u₂, u)
2 d(v, x)
g(v)
F(u₂)
g(u₂)
d(v₁, v)
d(v₂, v)
g(v₁)
g(v₂)
F(v₁)
F(v₂)


S
$g(y)$
$F(u_1)$
$g(u_1)$
$\gamma\, d(x,y)$
$g(x)$
$d(u_1,u)$
$2\, d(u,x)$
$g(u)$
$d(u_2,u)$
$2\, d(v,x)$
$g(v)$
$G(y) = \bigcap_{z,z_1,z_2 \in \mathcal{M}} H(z_1,z_2,z:y)$
$F(u_2)$
$g(u_2)$
$d(v_1,v)$
$d(v_2,v)$
$g(v_1)$
$g(v_2)$
$F(v_1)$
$F(v_2)$

We recall the structure of the set $H(z_1, z_2, z : y)$:



The proof relies on the following two auxiliary results.

Proposition 11.

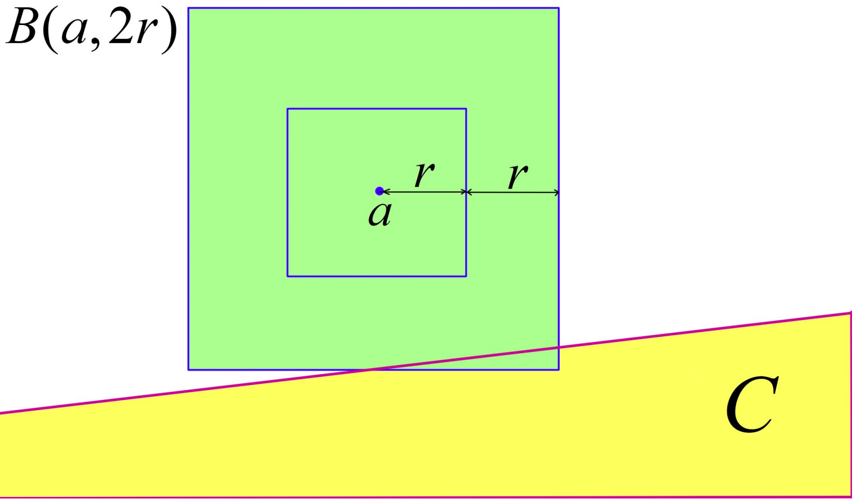
Let $C \subset Y$ be a convex set. Let $a \in Y$ and let $r > 0$. Suppose

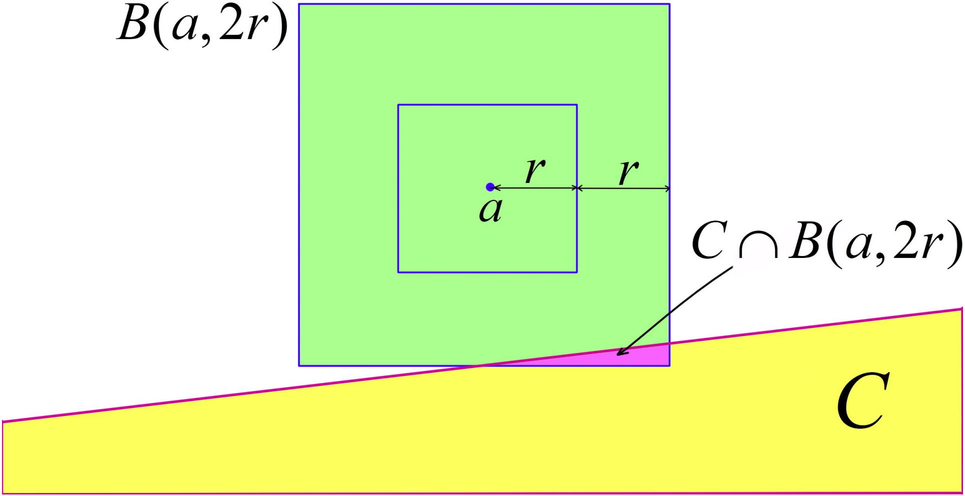
$$C \cap B(a, r) \neq \emptyset.$$

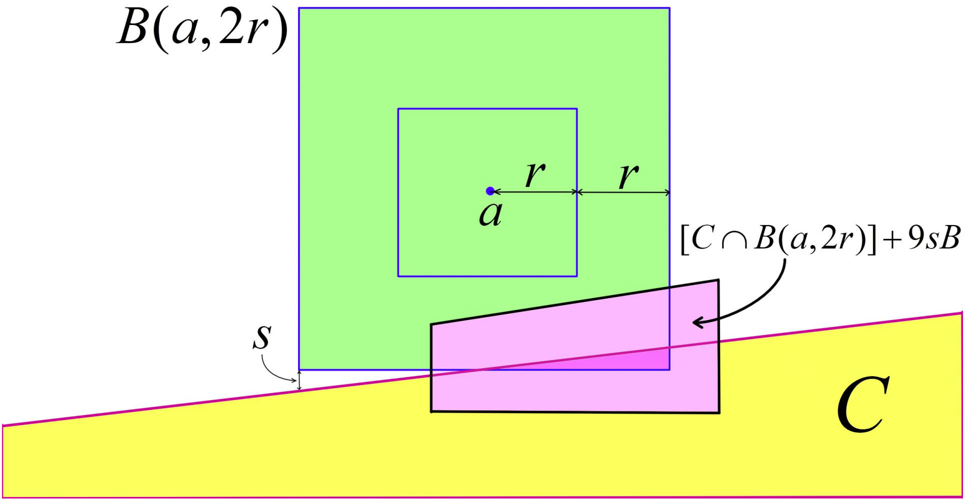
Then for every $s > 0$

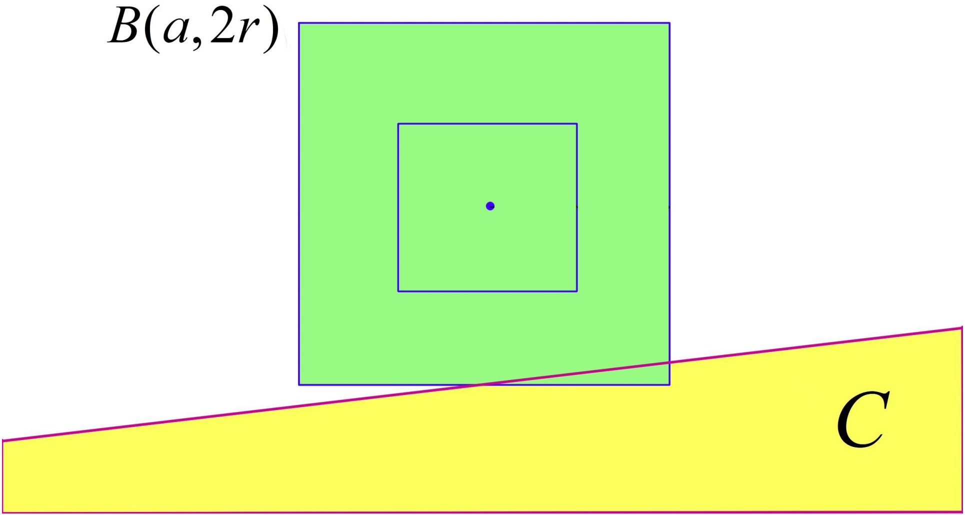
$$C \cap B(a, 2r) + 9s\, B \supset (C + sB) \cap (B(a, 2r) + sB).$$

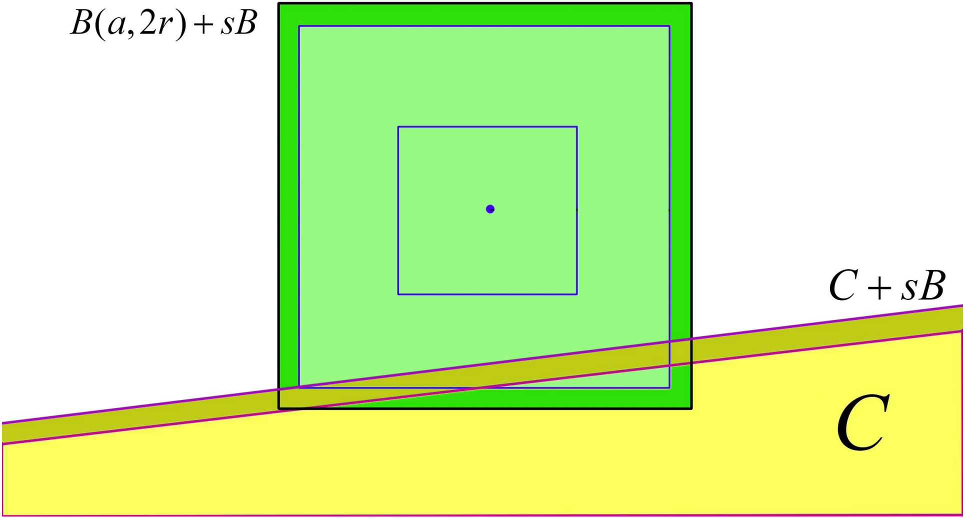
The next pictures illustrate the geometrical background of this imbedding.

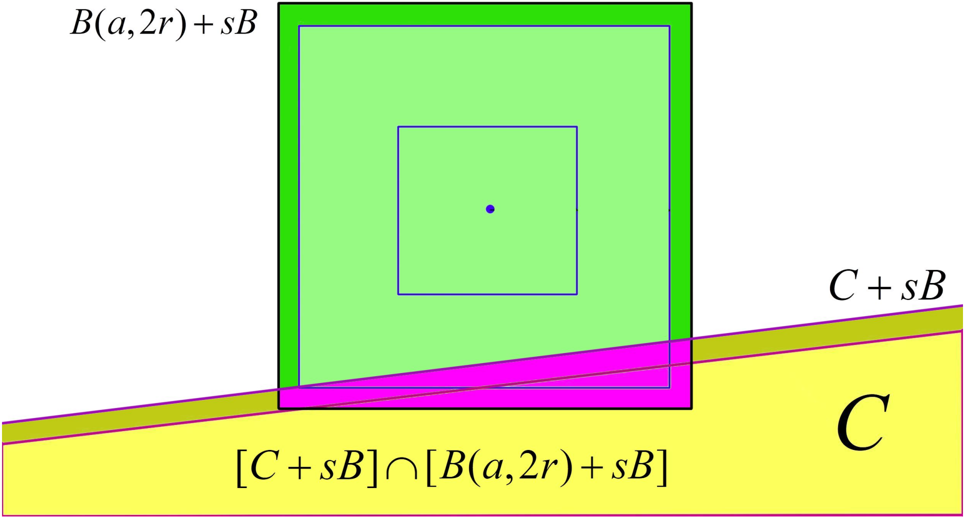

$B(a,2r)$
r
r
a
C

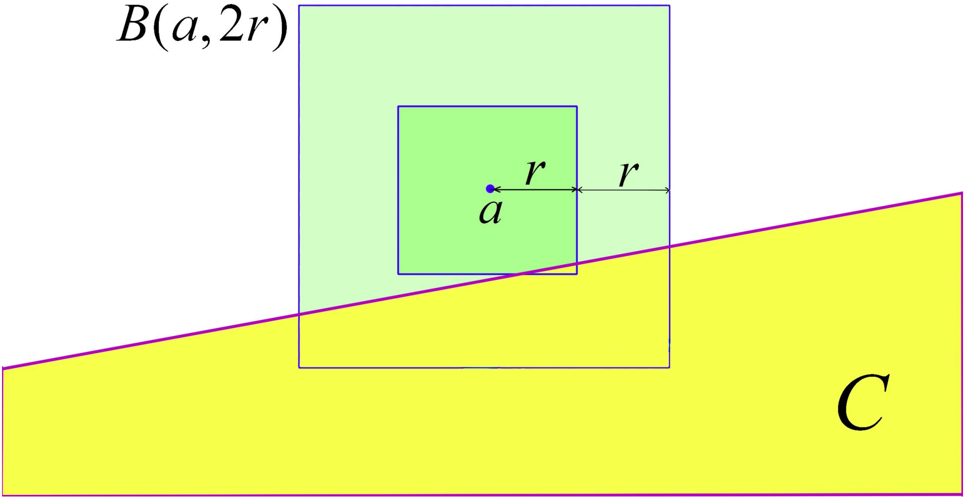

$B(a,2r)$
r
r
a
$C \cap B(a,2r)$
C

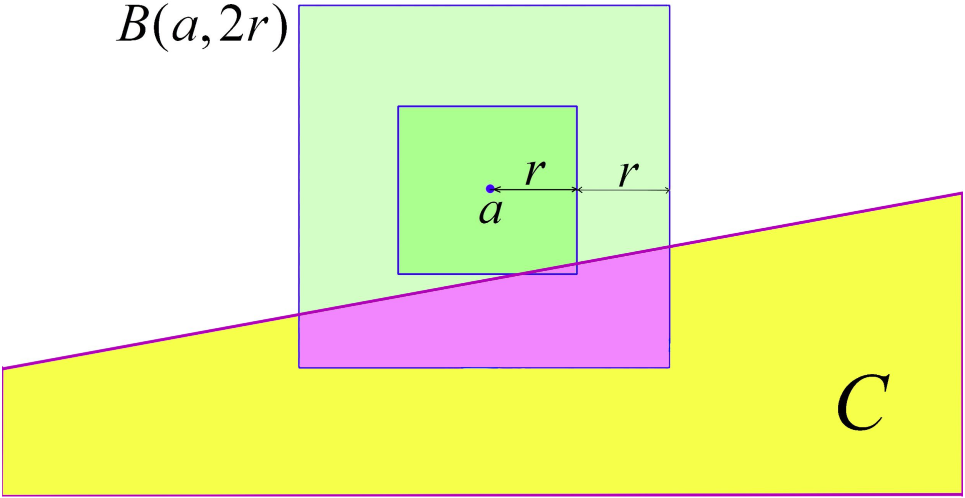

$B(a,2r)$
r
r
a
$[C \cap B(a,2r)] + 9sB$
s
C

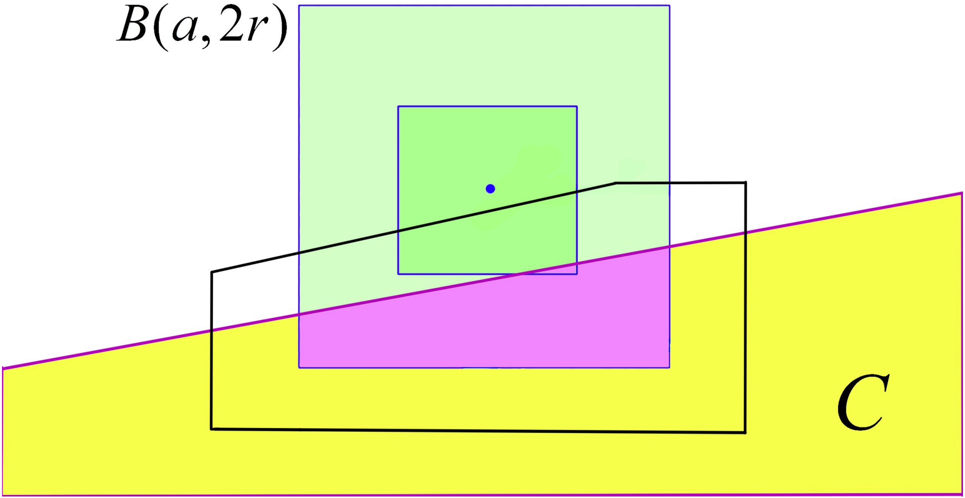

$B(a,2r)$
C

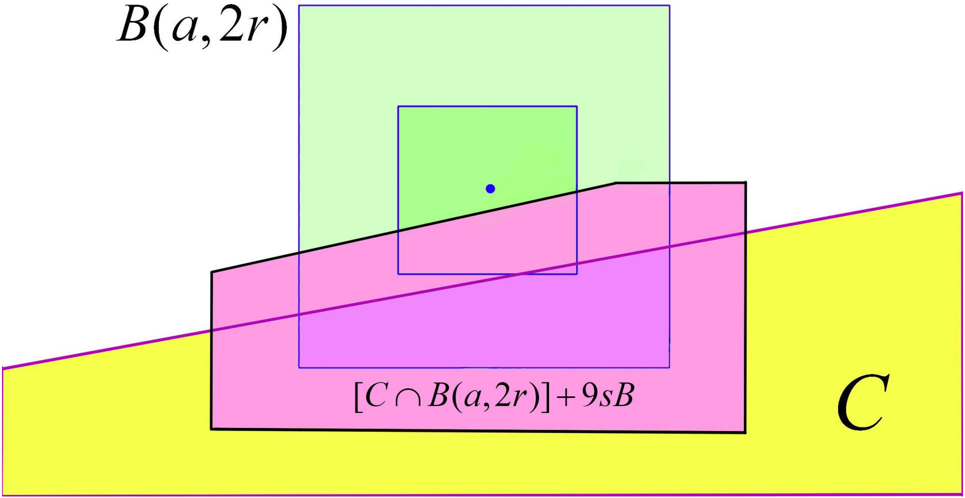

$B(a,2r)+sB$
$C+sB$
C

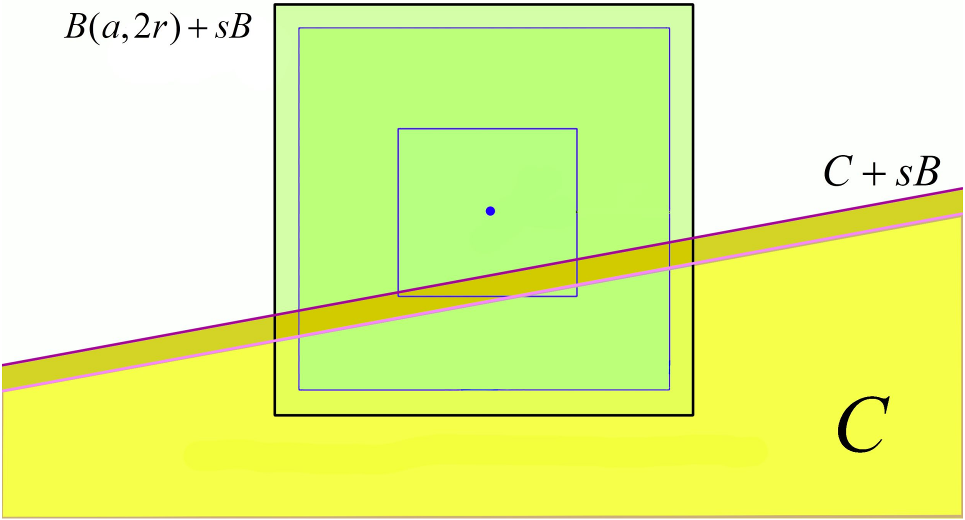

$B(a,2r)+sB$
$C+sB$
C
$[C+sB]\cap[B(a,2r)+sB]$

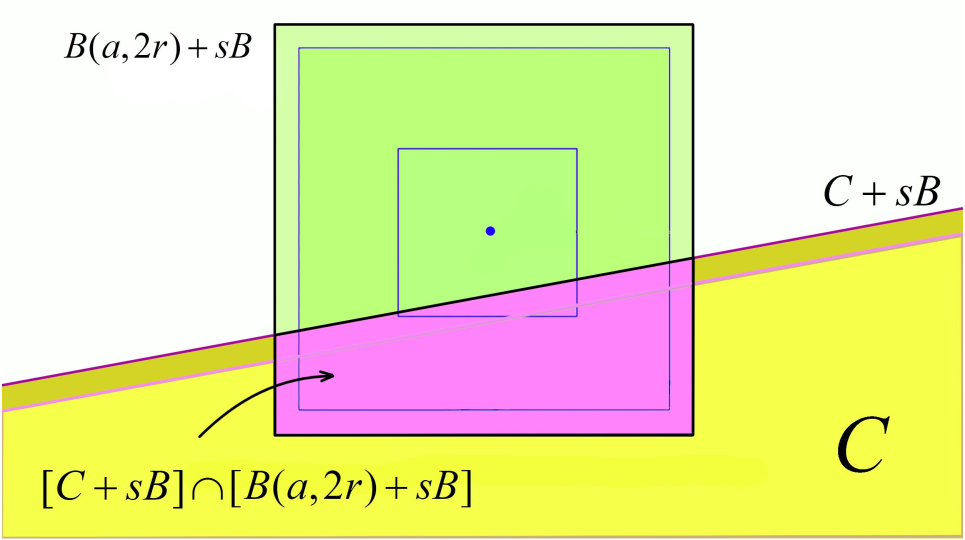

$B(a,2r)$
r
r
a
C

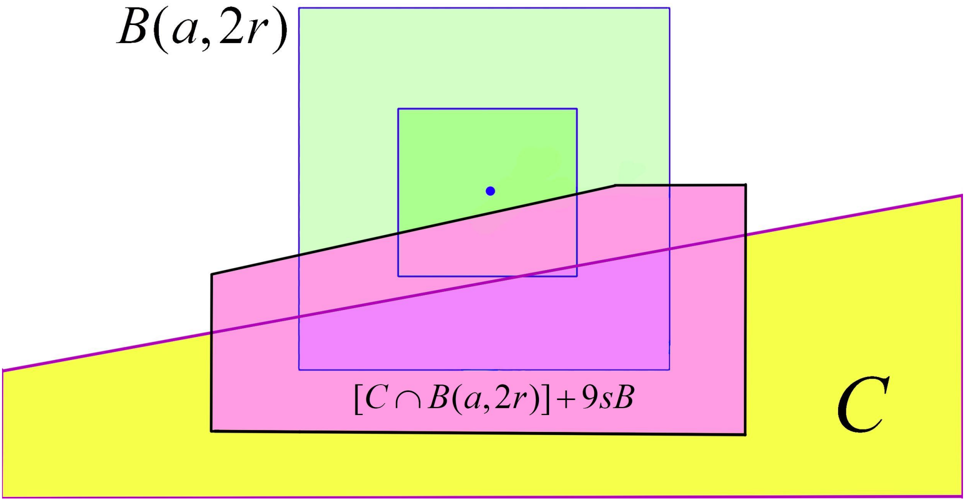

$B(a,2r)$
r
r
a
C


$B(a,2r)$
C


$B(a, 2r)$
$[C \cap B(a, 2r)] + 9sB$
C


$B(a, 2r) + sB$
$C + sB$
C


$B(a,2r)+sB$
$C+sB$
C
$[C+sB]\cap[B(a,2r)+sB]$


$B(a,2r)$
$[C \cap B(a,2r)] + 9sB$
C

Proposition 11 and Helly's Theorem in $\mathbb{R}^2$ imply the following result.

Proposition 12.

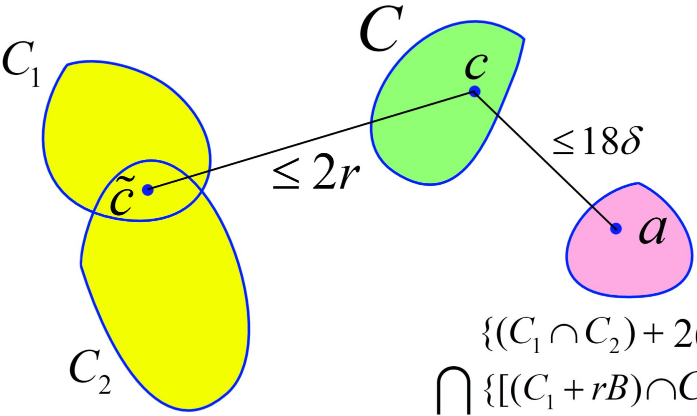
Let $C, C_1, C_2 \subset \mathbb{R}^2$ be convex subsets, and let $r > 0$. Let us assume that

$$C_1 \cap C_2 \cap (C + rB) \neq \emptyset.$$

Then for every $\delta > 0$

$$\{\,(C_1 \cap C_2) + 2rB\,\} \cap C + 18\delta B \supset$$

$$[(C_1 \cap C_2) + 2(r + \delta)B] \cap [((C_1 + rB) \cap C) + 2\delta B] \cap [((C_2 + rB) \cap C) + 2\delta B]$$



$$\{(C_1 \cap C_2) + 2(r+\delta)B\}$$
$$\bigcap \{[(C_1 + rB) \cap C] + 2\delta B\}$$
$$\bigcap \{[(C_2 + rB) \cap C] + 2\delta B\}$$

A Sketch of the Proof.

Let

$$a \in$$

$$[C_1 \cap C_2 + 2(r+\delta)B] \cap [(C_1 + rB) \cap C + 2\delta B] \cap [(C_2 + rB) \cap C + 2\delta B].$$

Using Helly's Theorem and the hypothesis of the proposition we prove that there exists a point $x \in \mathbb{R}^2$ such that

$$x \in C_1 \cap C_2 \cap (C + rB) \cap B(a, 2r + 2\delta)\,.$$

Hence, $x \in C + rB$ so that

$$B(x, r) \cap C \neq \emptyset\,.$$

Proposition 12 tells us that in this case

$$\begin{aligned} C \cap B(x, 2r) + 18\delta B \quad &\supset \quad [C + 2\delta B] \cap [B(x, 2r) + 2\delta B] \\ &= \quad [C + 2\delta B] \cap B(x, 2r + 2\delta)\,. \end{aligned}$$

Recall that

$$a \in [C_1 \cap C_2 + 2(r+\delta)B] \cap [(C_1 + rB) \cap C + 2\delta B] \cap [(C_2 + rB) \cap C + 2\delta B],$$

$$x \in C_1 \cap C_2 \cap (C + rB) \cap B(a, 2r + 2\delta)\,.$$

Then $x \in B(a, 2r+2\delta)$ so that $a \in B(x, 2r+2\delta)$.

Furthermore, $a \in [(C_1 + rB) \cap C] + 2\delta B \subset C + 2\delta B \quad \Longrightarrow$

$$(C + 2\delta B) \cap B(x, 2r + 2\delta) \ni a\,.$$

Hence,

$$C \cap B(x, 2r) + 18\delta B \supset [C + 2\delta B] \cap B(x, 2r + 2\delta) \ni a.$$

But $x \in C_1 \cap C_2$ which proves the required inclusion

$$[(C_1 \cap C_2) + 2rB] \cap C + 18\delta B \ni a\,. \qquad \square$$

We return to the proof of the imbedding

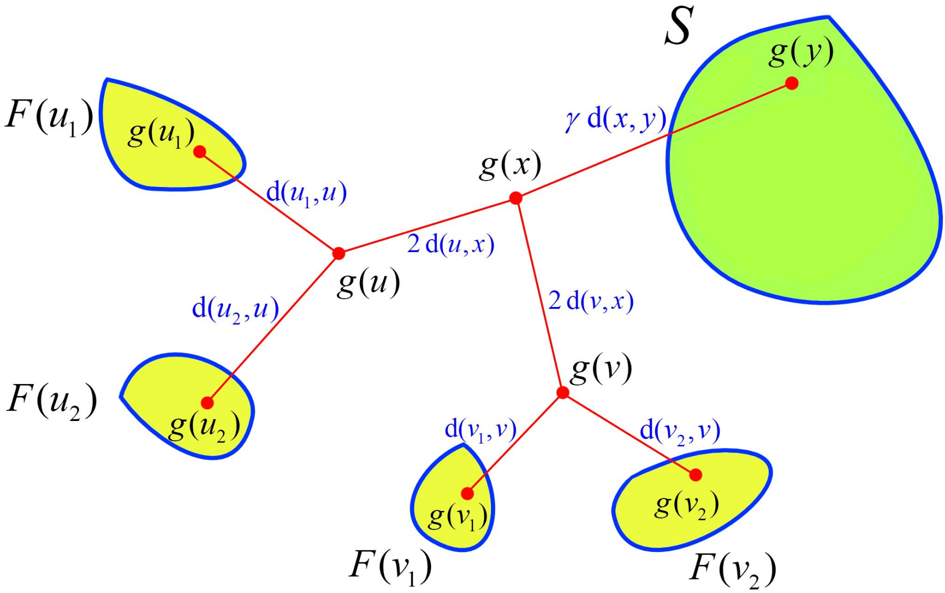
$$S = \Big[H(u_1, u_2, u : x) \bigcap H(v_1, v_2, v : x)\Big] + \gamma\, \mathrm{d}(x, y)\, B \supset \bigcap_{z, z_1, z_2 \in \mathcal{M}} H(z_1, z_2, z : y).$$

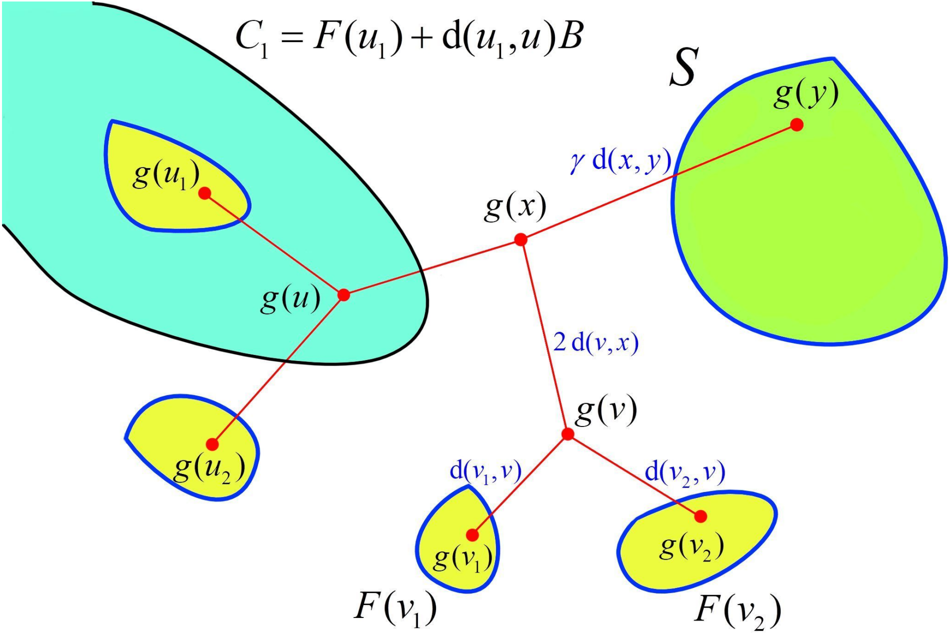
We recall that

$$H(u_1, u_2, u : x) = \Big\{\big[F(u_1) + \mathrm{d}(u_1, z)B\big] \bigcap \big[F(u_2) + \mathrm{d}(u_2, z)B\big]\Big\} + 2\,\mathrm{d}(u, x)B$$

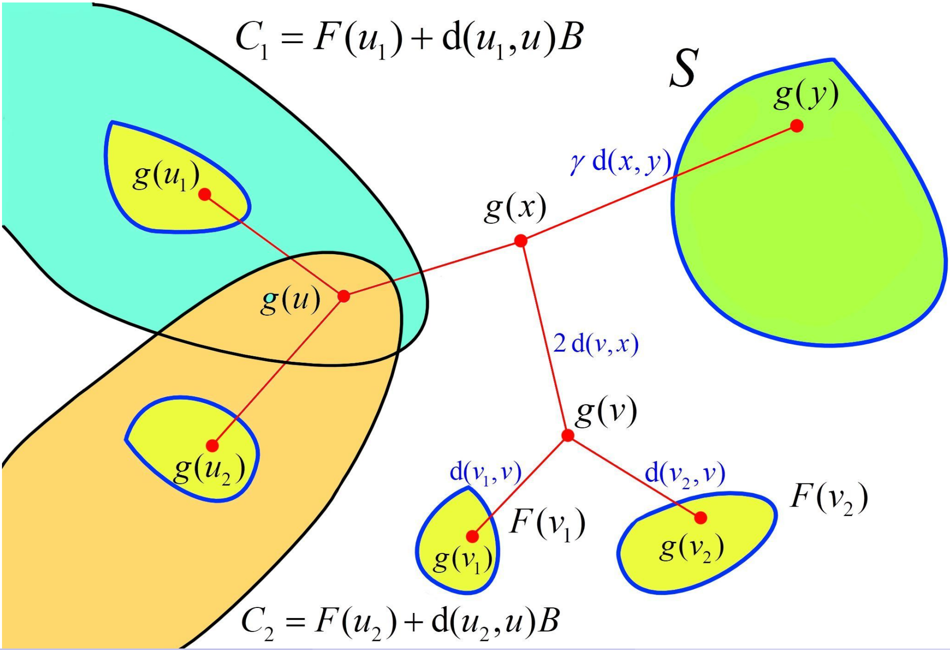
and

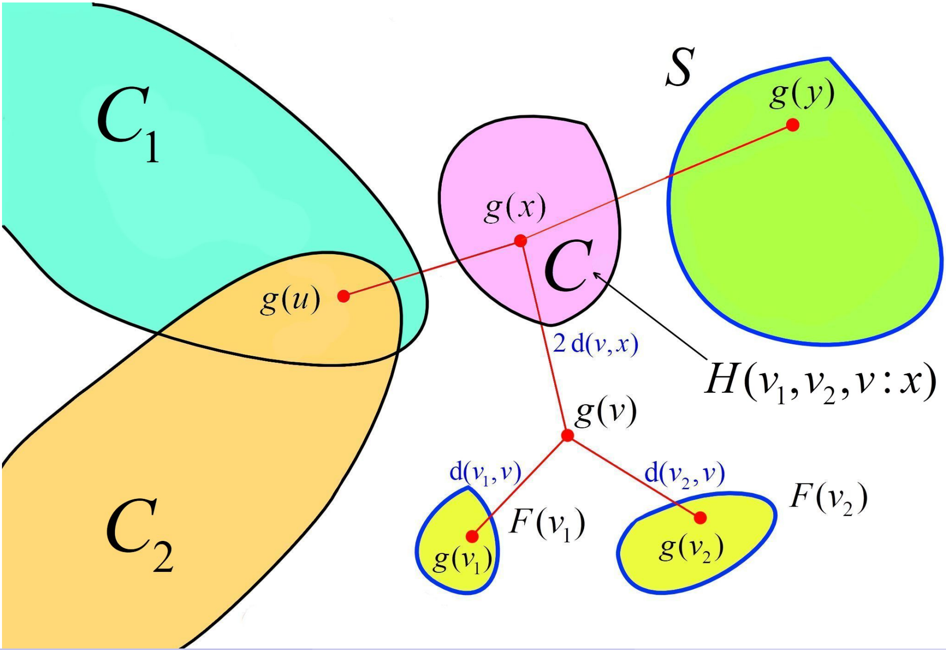
$$H(v_1, v_2, v : x) = \Big\{\big[F(v_1) + \mathrm{d}(v_1, v)B\big] \bigcap \big[F(v_2) + \mathrm{d}(v_2, v)B\big]\Big\} + 2\,\mathrm{d}(v, x)B.$$

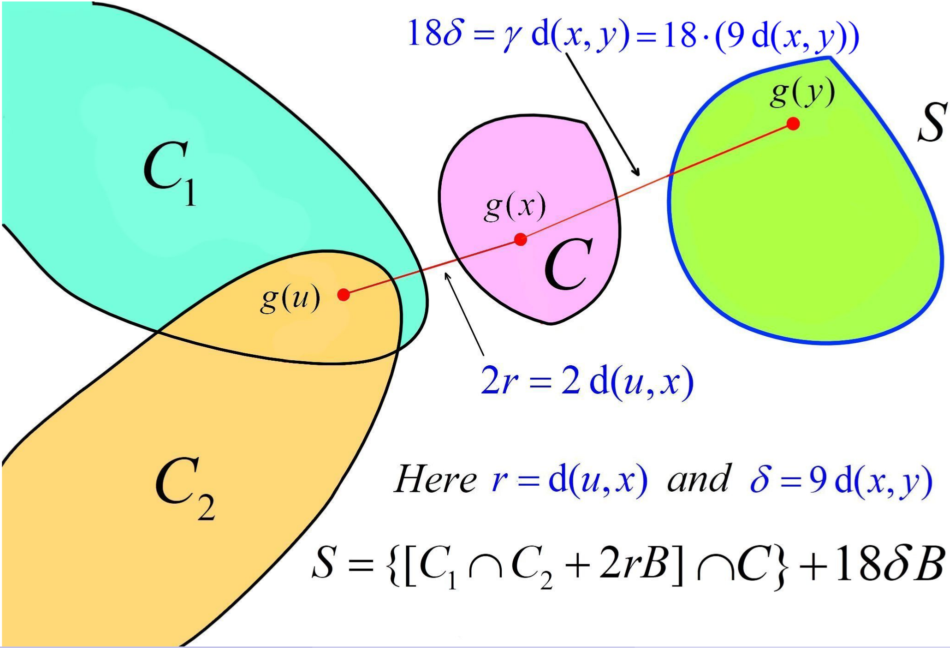

S
$g(y)$
$F(u_1)$
$g(u_1)$
$\gamma\, d(x,y)$
$g(x)$
$d(u_1,u)$
$2\, d(u,x)$
$g(u)$
$d(u_2,u)$
$2\, d(v,x)$
$g(v)$
$F(u_2)$
$g(u_2)$
$d(v_1,v)$
$d(v_2,v)$
$g(v_1)$
$g(v_2)$
$F(v_1)$
$F(v_2)$



$C_1 = F(u_1) + d(u_1, u)B$
S
$g(y)$
$\gamma\, d(x, y)$
$g(u_1)$
$g(x)$
$g(u)$
$2\, d(v, x)$
$g(v)$
$g(u_2)$
$d(v_1, v)$
$d(v_2, v)$
$g(v_1)$
$g(v_2)$
$F(v_1)$
$F(v_2)$


$C_1 = F(u_1) + \mathrm{d}(u_1, u)B$
S
$g(y)$
$\gamma\, \mathrm{d}(x, y)$
$g(u_1)$
$g(x)$
$g(u)$
$2\, \mathrm{d}(v, x)$
$g(v)$
$g(u_2)$
$\mathrm{d}(v_1, v)$
$\mathrm{d}(v_2, v)$
$F(v_2)$
$F(v_1)$
$g(v_2)$
$g(v_1)$
$C_2 = F(u_2) + \mathrm{d}(u_2, u)B$


C_1
C_2
S
$g(y)$
$g(x)$
C
$g(u)$
$2\,d(v,x)$
$H(v_1,v_2,v:x)$
$g(v)$
$d(v_1,v)$
$d(v_2,v)$
$F(v_2)$
$F(v_1)$
$g(v_1)$
$g(v_2)$



Here $r = d(u,x)$ *and* $\delta = 9\, d(x,y)$

$$S = \{[C_1 \cap C_2 + 2rB] \cap C\} + 18\delta B$$

To apply Proposition 12 to the set S we have to check that

$$C_1 \cap C_2 \cap (C + rB) \neq \emptyset\,.$$

We know that the restriction $F|_{\mathcal{B}}$ of F to the set

$$\mathcal{B} = \{u_1, u_2, u, v_1, v_2, v, x, \}$$

has a Lipschitz selection $f : \mathcal{B} \to \mathbb{R}^2$ with $\|f\|_{\mathrm{Lip}(\mathcal{B},\mathbb{R}^2)} \leq 1$.

Then,

$$C_1 \cap C_2 \cap (C + rB) \ni f(u)$$

proving that the hypothesis of Proposition 12 holds.

By this proposition,

$$S = (C_1 \cap C_2 + 2rB) \cap C + 18\delta\, B \supset$$

$$[(C_1 \cap C_2) + 2(r+\delta)B] \cap [((C_1 + rB) \cap C) + 2\delta B] \cap [((C_2 + rB) \cap C) + 2\delta B]$$

$$= A_1 \cap A_2 \cap A_3\,.$$

Prove that

$$A_1 = (C_1 \cap C_2) + 2(r+\delta)B \supset G(y),$$

$$A_2 = ((C_1 + rB) \cap C) + 2\delta B \supset G(y),$$

and

$$A_3 = ((C_2 + rB) \cap C) + 2\delta B \supset G(y).$$

Prove that

$$A_1 = (C_1 \cap C_2) + 2(r+\delta)B \supset H(u_1, u_2, u : y).$$

Recall that

$$A_1 = (C_1 \cap C_2) + 2(r+\delta)B =$$

$$\{F(u_1) + \mathrm{d}(u_1, u)B\} \cap \{F(u_2) + \mathrm{d}(u_2, u)B\} + 2(\mathrm{d}(u, x) + 9\,\mathrm{d}(x, y))B\,.$$

By the triangle inequality,

$$\mathrm{d}(u, x) + 9\,\mathrm{d}(x, y) \geq \mathrm{d}(u, x) + \mathrm{d}(x, y) \geq \mathrm{d}(u, y)$$

so that

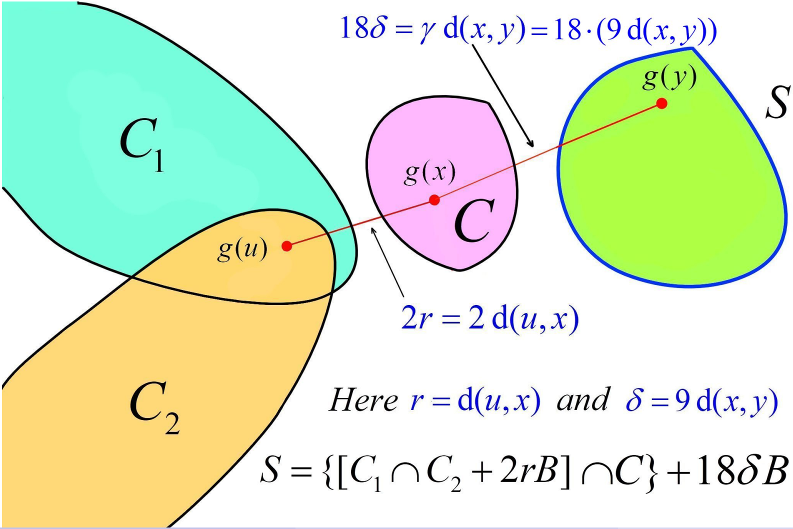
$$A_1 = (C_1 \cap C_2) + 2(r+\delta)B \supset$$

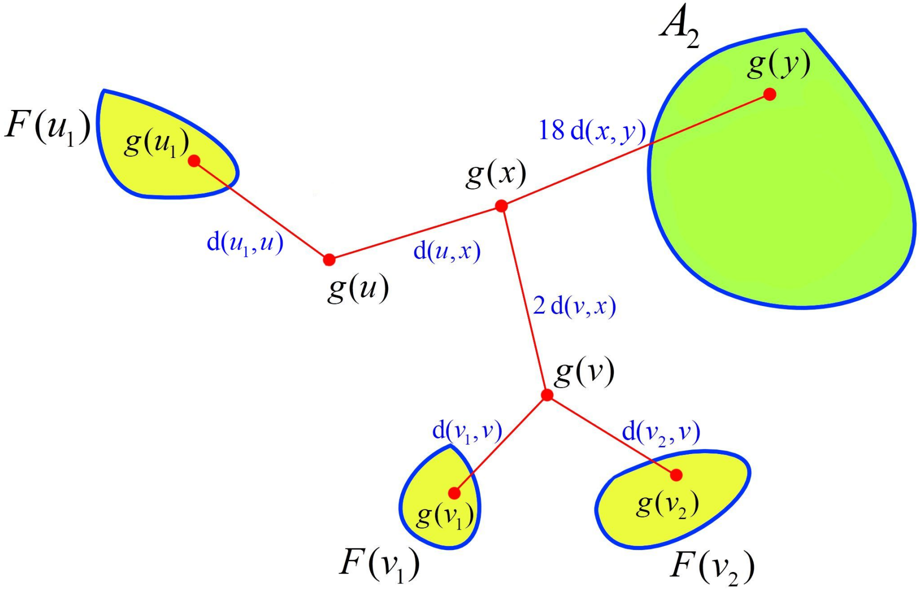
$$\{F(u_1) + \mathrm{d}(u_1, u)B\} \cap \{F(u_2) + \mathrm{d}(u_2, u)B\} + 2\,\mathrm{d}(u, y)B$$

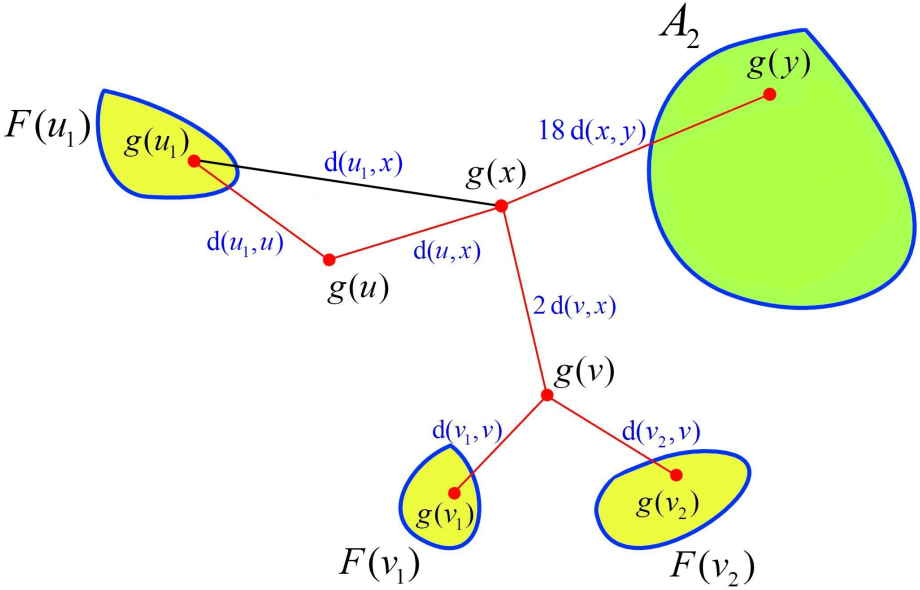
$$= H(u_1, u_2, u : y) \supset G(y)\,.$$

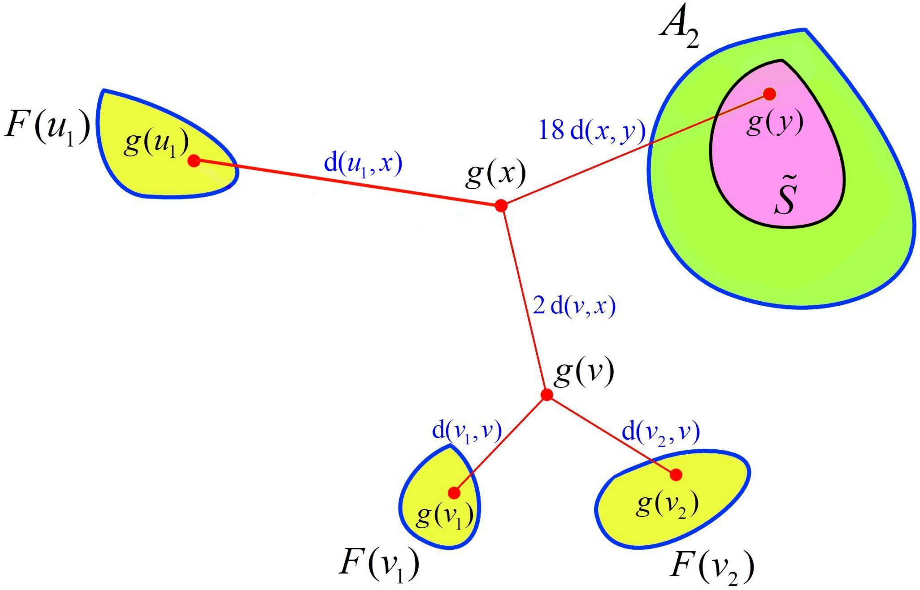
Prove that

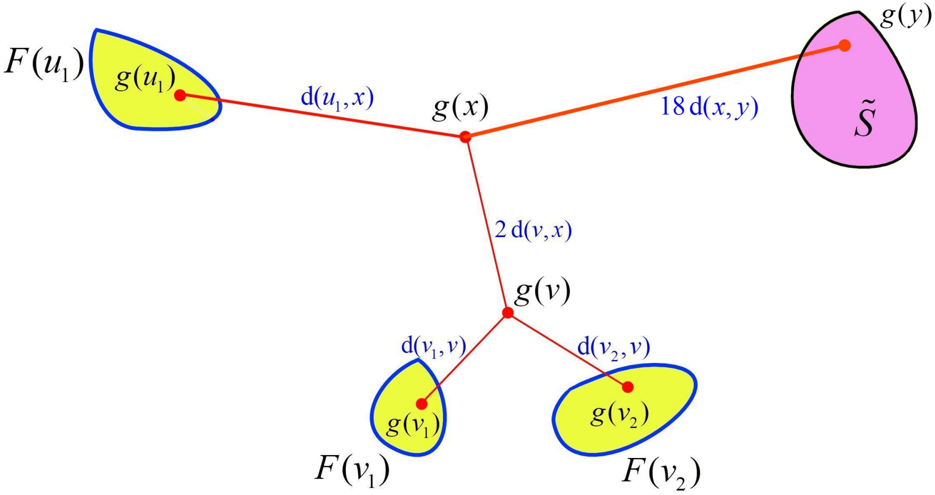
$$A_2 = ((C_1 + rB) \cap C) + 2\delta B \supset G(y).$$

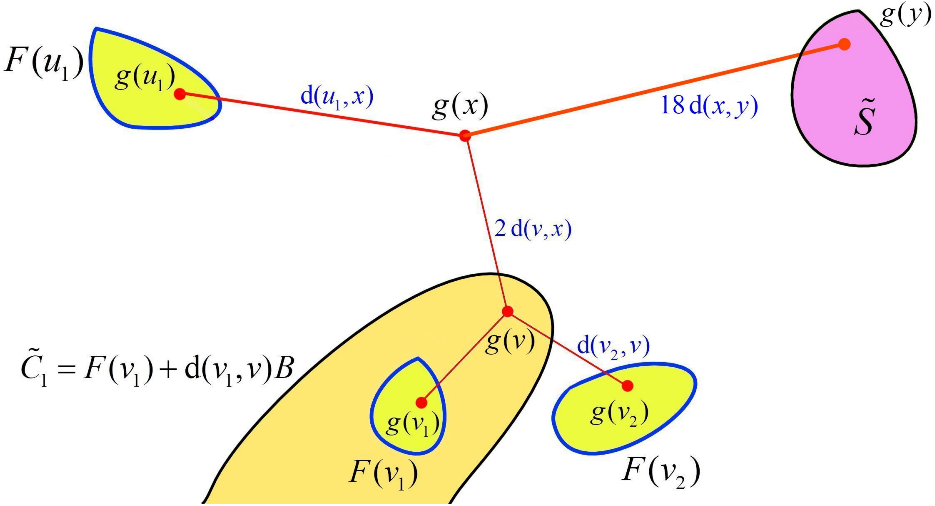



A_2
$g(y)$
$F(u_1)$
$g(u_1)$
$18\,d(x,y)$
$g(x)$
$d(u_1,u)$
$d(u,x)$
$g(u)$
$2\,d(v,x)$
$g(v)$
$d(v_1,v)$
$d(v_2,v)$
$g(v_1)$
$g(v_2)$
$F(v_1)$
$F(v_2)$

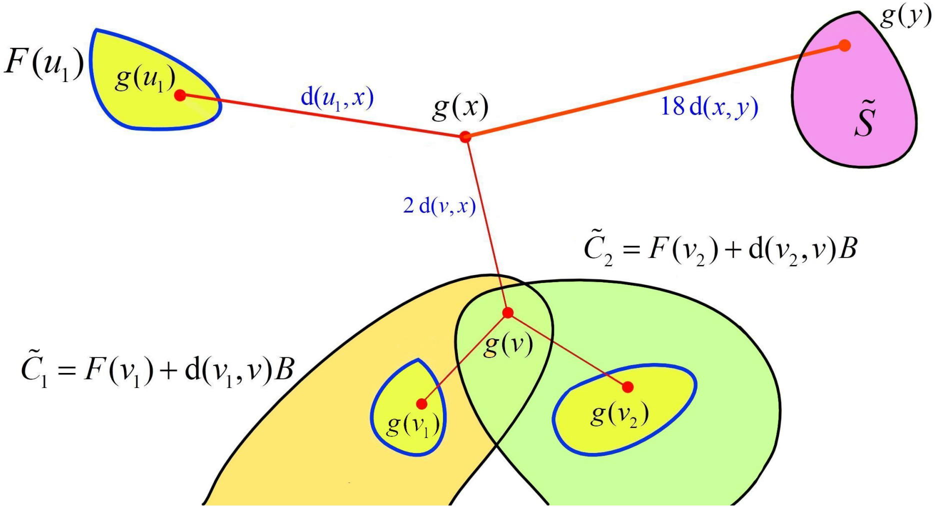

A_2
$g(y)$
$F(u_1)$
$g(u_1)$
$18\,d(x,y)$
$d(u_1,x)$
$g(x)$
$d(u_1,u)$
$d(u,x)$
$g(u)$
$2\,d(v,x)$
$g(v)$
$d(v_1,v)$
$d(v_2,v)$
$g(v_1)$
$g(v_2)$
$F(v_1)$
$F(v_2)$


A_2
$F(u_1)$
$g(u_1)$
$\mathrm{d}(u_1, x)$
$g(x)$
$18\,\mathrm{d}(x, y)$
$g(y)$
$\tilde{S}$
$2\,\mathrm{d}(v, x)$
$g(v)$
$\mathrm{d}(v_1, v)$
$\mathrm{d}(v_2, v)$
$g(v_1)$
$g(v_2)$
$F(v_1)$
$F(v_2)$

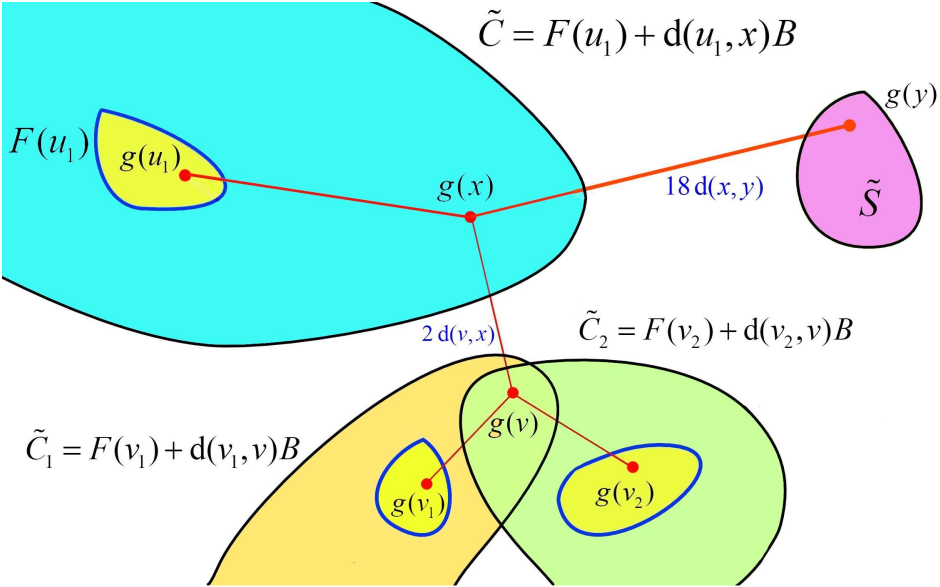

$g(y)$
$F(u_1)$
$g(u_1)$
$d(u_1, x)$
$g(x)$
$18\,d(x, y)$
$\tilde{S}$
$2\,d(v, x)$
$g(v)$
$d(v_1, v)$
$d(v_2, v)$
$g(v_1)$
$g(v_2)$
$F(v_1)$
$F(v_2)$



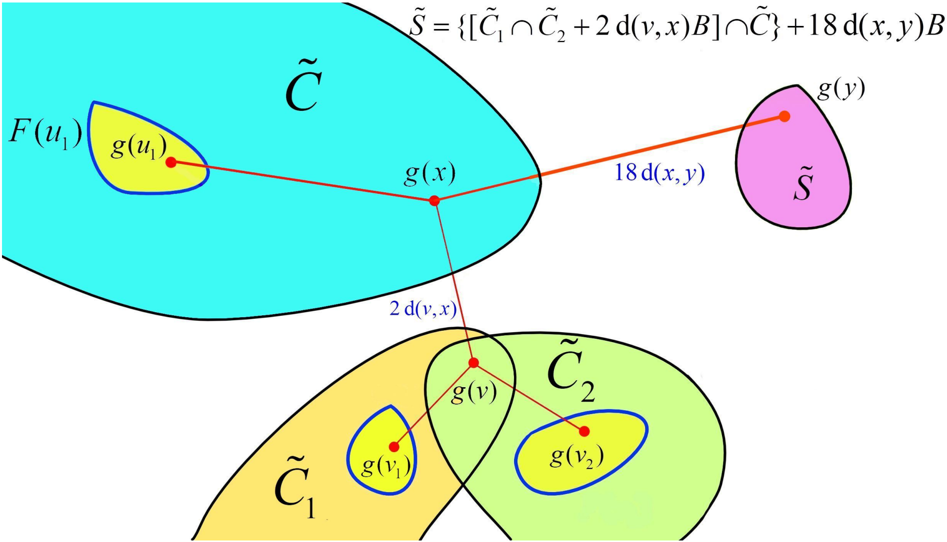
$F(u_1)$
$g(u_1)$
$d(u_1, x)$
$g(x)$
$18\, d(x, y)$
$g(y)$
$\tilde{S}$
$2\, d(v, x)$
$g(v)$
$d(v_2, v)$
$\tilde{C}_1 = F(v_1) + d(v_1, v)B$
$g(v_1)$
$F(v_1)$
$g(v_2)$
$F(v_2)$



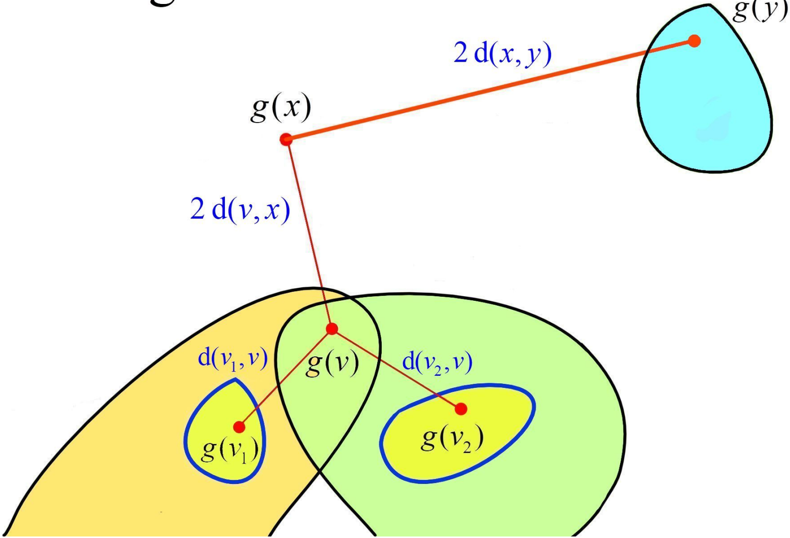
$F(u_1)$
$g(u_1)$
$\mathrm{d}(u_1, x)$
$g(x)$
$18\,\mathrm{d}(x, y)$
$g(y)$
$\tilde{S}$
$2\,\mathrm{d}(v, x)$
$\tilde{C}_2 = F(v_2) + \mathrm{d}(v_2, v)B$
$g(v)$
$\tilde{C}_1 = F(v_1) + \mathrm{d}(v_1, v)B$
$g(v_1)$
$g(v_2)$


$\tilde{C} = F(u_1) + \mathrm{d}(u_1, x)B$
$g(y)$
$F(u_1)$
$g(u_1)$
$g(x)$
$18\,\mathrm{d}(x, y)$
$\tilde{S}$
$2\,\mathrm{d}(v, x)$
$\tilde{C}_2 = F(v_2) + \mathrm{d}(v_2, v)B$
$g(v)$
$\tilde{C}_1 = F(v_1) + \mathrm{d}(v_1, v)B$
$g(v_1)$
$g(v_2)$

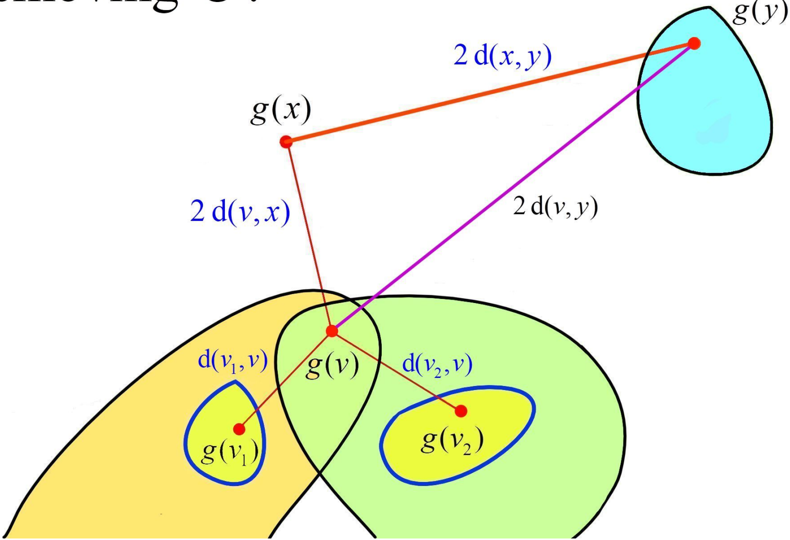
$$\tilde{S} = \{[\tilde{C}_1 \cap \tilde{C}_2 + 2\,\mathrm{d}(v,x)B] \cap \tilde{C}\} + 18\,\mathrm{d}(x,y)B$$



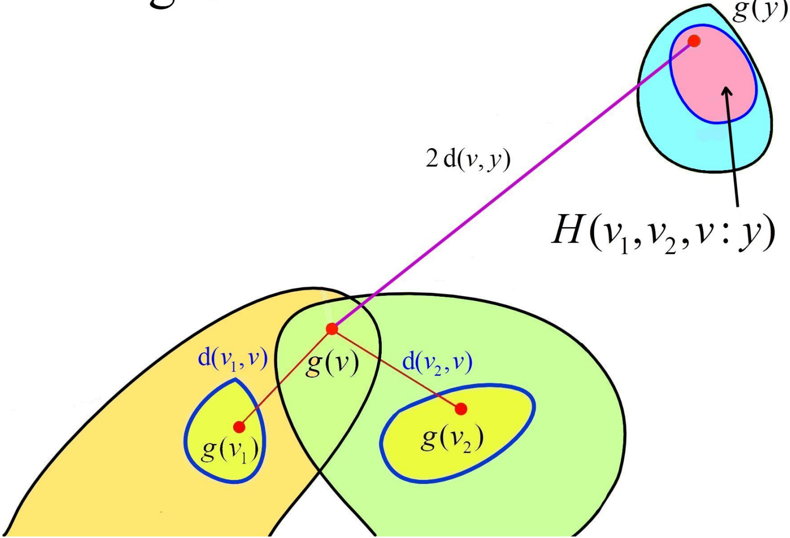
Removing $\tilde{C}$:



Removing $\tilde{C}$:



Removing $\tilde{C}$:



Applying Proposition 12 we obtain the required inclusion

$$A_2 \supset H(v_1, v_2, v : y) \cap H(u_1, v_1, x : y) \cap H(u_1, v_2, x : y) \supset G(y).$$

In the same fashion we show that

$$A_3 = [((C_2 + rB) \cap C) + 2\delta B] \supset G(y)$$

proving the required imbedding

$$G(x) + \gamma\, \mathrm{d}(x, y)B \supset G(y)$$

with $\gamma = 2 \cdot 9^2 = 162$.

By interchanging the roles of x and y we obtain also

$$G(y) + \gamma\, \mathrm{d}(x, y)B \supset G(x).$$

Hence,

$$\mathrm{d_H}(G(x), G(y)) \leq \gamma\, \mathrm{d}(x, y) = 2^6\, \gamma\rho(x, y), \qquad x, y \in \mathcal{M},$$

proving that the set-valued mapping G is a $2^6\,\gamma$-core of F. □

7. Lipschitz Selection in $\mathbb{R}^2$: an Algorithm.

The proof of Theorem 6 provides an efficient algorithm for constructing of an almost optimal Lipschitz selection for any set-valued mapping $F : \mathcal{M} \to \mathcal{K}_2(\mathbb{R}^2)$ satisfying the hypothesis of the Finiteness Principle.

- $Y = \ell^2_\infty = (\mathbb{R}^2, \|\cdot\|)$, where $\|x\| = \max\{|x_1|, |x_2|\}$ for $x = (x_1, x_2) \in \mathbb{R}^2$;
- $Q_0 = [-1, 1] \times [-1, 1]$;
- "box" or "rectangle" - a rectangle in $\mathbb{R}^2$ with sides parallel to the coordinate axes;
- $\mathcal{R}(\mathbb{R}^2)$ - the family of all "boxes" in $\mathbb{R}^2$.
- Given $G \subset \mathbb{R}^2$ we let $H[G]$ denote the smallest box containing G:

$$H[G] = \bigcap \left\{ \Pi = [a, b] \times [c, d] \subset \mathbb{R}^2 : \ \Pi \supset G \right\}$$

Let $(\mathcal{M},\rho)$ be a pseudometric space, and let $F:\mathcal{M}\to\mathcal{K}_2(\mathbb{R}^2)$ be a set-valued mapping satisfying the following condition:

There exists a constant $\alpha>0$ such that for every subset $\mathcal{M}'\subset\mathcal{M}$ with $\#\mathcal{M}'\le 4$ the restriction $F|_{\mathcal{M}'}$ has a Lipschitz selection $f_{\mathcal{M}'}:\mathcal{M}'\to\mathbb{R}^2$ with the Lipschitz seminorm

$$\|f_S\|_{\mathrm{Lip}(\mathcal{M}',\mathbb{R}^2)}\le\alpha.$$

STEP 1. We construct a $2^6\alpha$-balanced refinement of F:

$$F^{[1]}(x)=\bigcap_{y\in\mathcal{M}}\left[F(y)+2^6\alpha\,\rho(x,y)\,Q_0\right],\quad x\in\mathcal{M}.$$

STEP 2. We construct a $2^7\alpha$-balanced refinement of $F^{[1]}$:

$$F^{[2]}(x)=\bigcap_{y\in\mathcal{M}}\left[F^{[1]}(y)+2^7\alpha\,\rho(x,y)\,Q_0\right],\quad x\in\mathcal{M}.$$

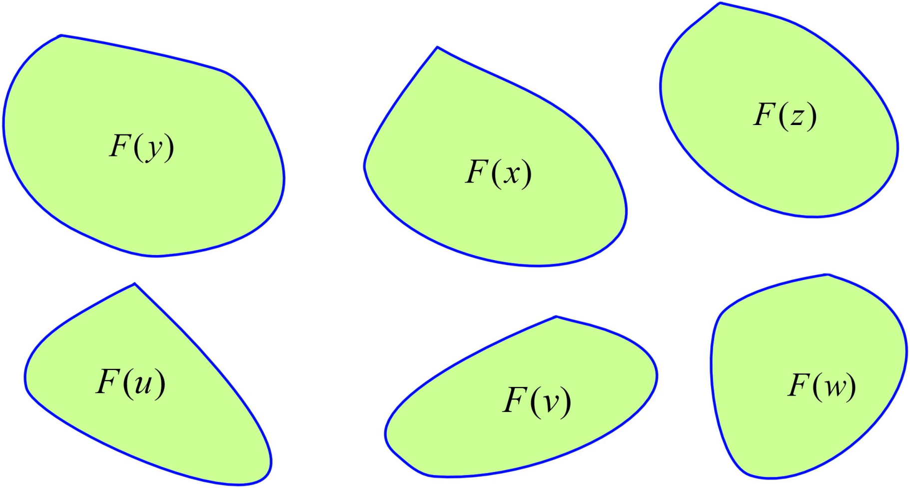
STEP 3. We construct a set-valued mapping $\mathcal{H}_F : \mathcal{M} \to \mathcal{R}(\mathbb{R}^2)$ which to every $x \in \mathcal{M}$ assigns the smallest box containing $F^{[2]}(x)$:

$$\mathcal{H}_F(x) = H\left[F^{[2]}(x)\right], \quad x \in \mathcal{M}.$$

STEP 4. We define a Lipschitz selection $f : \mathcal{M} \to \mathbb{R}^2$ of F by

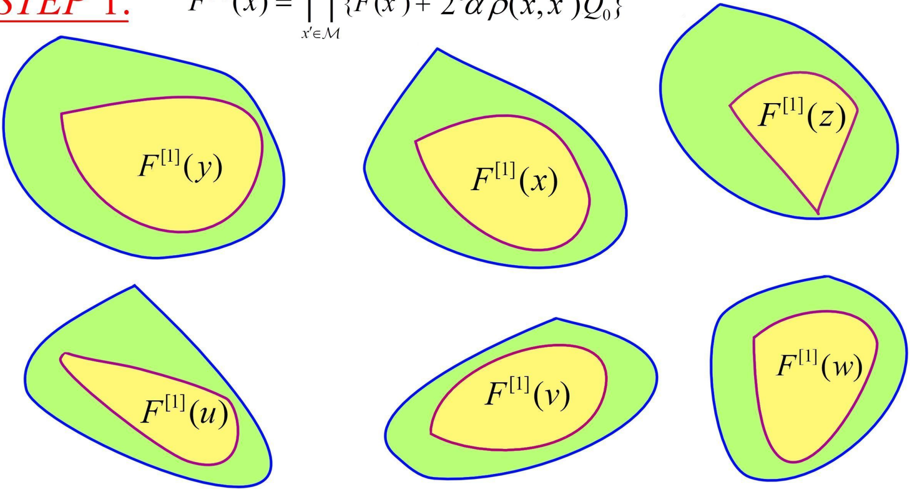
$$f(x) = \text{center}\,(\mathcal{H}_F(x)) = \text{center}\left(H\left[F^{[2]}(x)\right]\right), \quad x \in \mathcal{M}.$$

Here given a rectangle $P \in \mathcal{R}(\mathbb{R}^2)$ we let $\text{center}\,(P)$ denote the center of P.

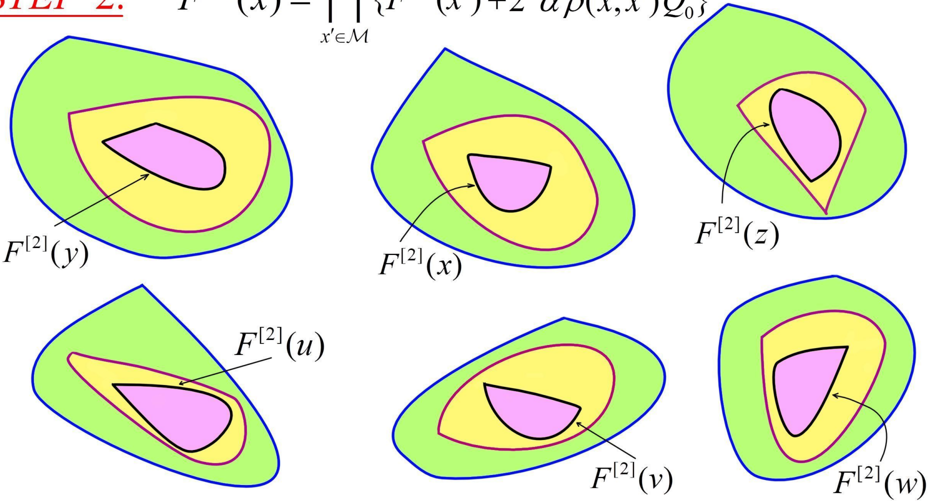


$F(y)$
$F(x)$
$F(z)$
$F(u)$
$F(v)$
$F(w)$

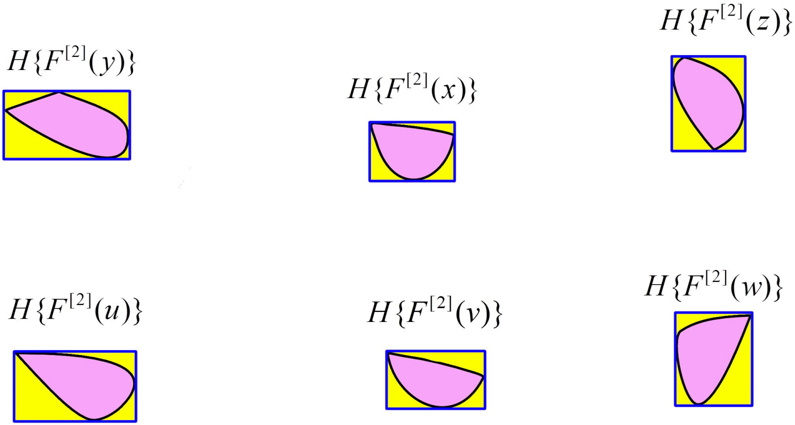
$\underline{STEP\ 1.}$ $$F^{[1]}(x) = \bigcap_{x' \in \mathcal{M}} \{F(x') + 2^6 \alpha\, \rho(x,x') Q_0\}$$



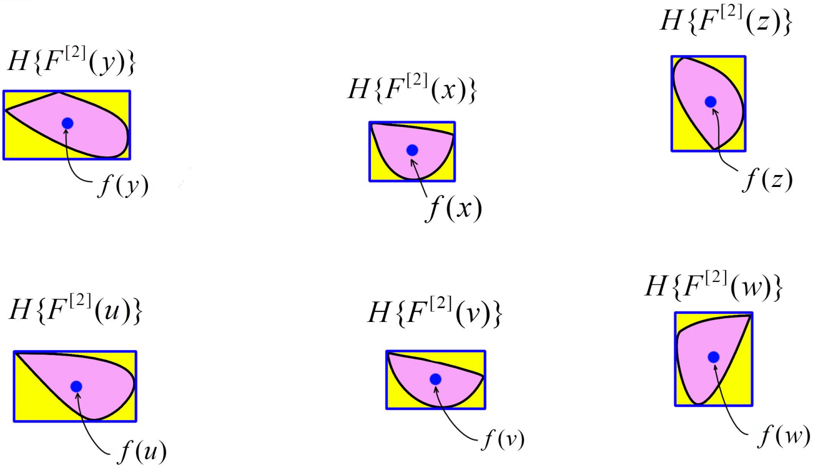
STEP 2. $F^{[2]}(x) = \bigcap_{x' \in \mathcal{M}} \{F^{[1]}(x') + 2^7 \alpha \rho(x, x') Q_0\}$



STEP 3. $H\{F^{[2]}(x)\}$ - the smallest box containing $F^{[2]}(x)$



STEP 4. $f(x)$ - the center of the box $H\{F^{[2]}(x)\}$



The following statement justifies STEP 3 and STEP 4 of the Algorithm.

Statement 14.

(1) Let $G \subset \mathbb{R}^2$ be a convex compact set. Then $\text{center}\,(H(G)) \in G$.

(2) Let $G_1, G_2 \subset \mathbb{R}^2$ be convex compact sets. Then

$$\text{d}_{\text{H}}(H[G_1], H[G_2]) \le \text{d}_{\text{H}}(G_1, G_2).$$

(3) For every two boxes $P_1, P_2 \in \mathcal{R}(\mathbb{R}^2)$ we have

$$\|\,\text{center}\,(P_1) - \text{center}\,(P_2)\| \le \text{d}_{\text{H}}(P_1, P_2).$$

(Recall that $\mathbb{R}^2$ is equipped with the ℓ^2_∞-norm.)

We know that the set-valued mapping $F^{[2]} : \mathcal{M} \to \mathcal{K}_2$ is a γ-core of F with $\gamma = 2^{14}\alpha$, i.e.,

$$\text{d}_{\text{H}}(F^{[2]}(x), F^{[2]}(y)) \le \gamma\,\rho(x,y), \qquad x, y \in \mathcal{M}.$$

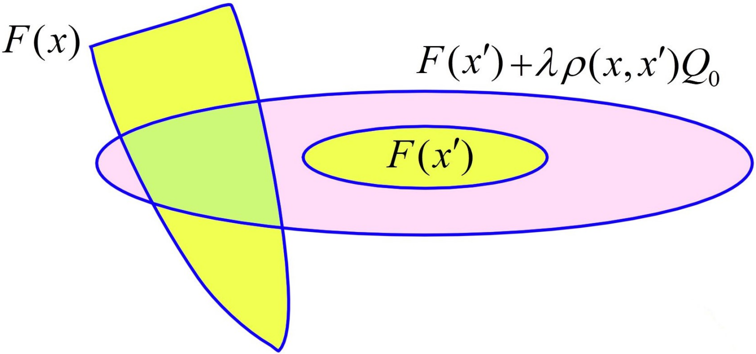
Combining this inequality with Statement 14 we conclude that f is a Lipschitz selection of F with $\|f\|_{\text{Lip}(\mathcal{M},\mathbb{R}^2)} \le \gamma$.

8. Criterions for Lipschitz Selections in $\mathbb{R}^2$

Let $Y = \ell^2_\infty$, and let $F : \mathcal{M} \to \mathcal{K}(\mathbb{R}^2)$ be a set valued mapping.

Given $\lambda > 0$ and $x, x' \in \mathcal{M}$, let

$$\mathcal{R}_\lambda[x, x' : F] = H[F(x) \cap \{F(x') + \lambda\,\rho(x, x')\, Q_0\}].$$



8. Criterions for Lipschitz Selections in $\mathbb{R}^2$

Let $Y = \ell^2_\infty$, and let $F : \mathcal{M} \to \mathcal{K}(\mathbb{R}^2)$ be a set valued mapping. Given $\lambda > 0$ and $x, x' \in \mathcal{M}$, let

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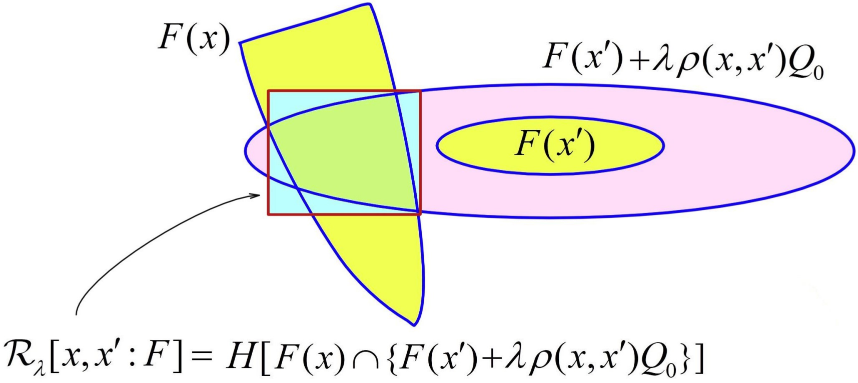


Theorem 15 (Sh. [2002])

A set-valued mapping $F : \mathcal{M} \to \mathcal{K}(\mathbb{R}^2)$ has a Lipschitz selection if and only if $\exists\, \lambda > 0$ such that:

(i) $\mathcal{R}_\lambda[x, x' : F] \neq \emptyset$ for every $x, x' \in \mathcal{M}$;

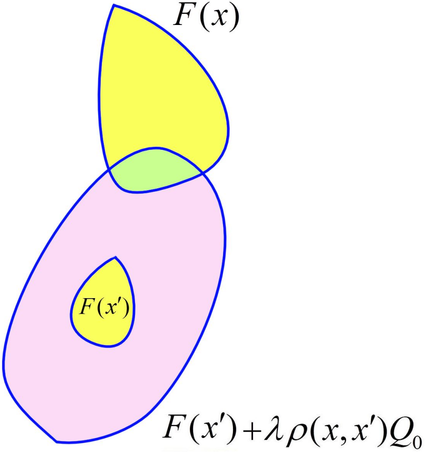
(ii) For every $x, x', y, y' \in \mathcal{M}$ the following inequality

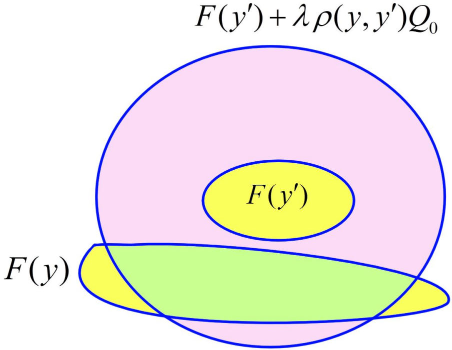
$$\operatorname{dist}\left(\mathcal{R}_\lambda[x, x' : F], \mathcal{R}_\lambda[y, y' : F]\right) \le \lambda\, \rho(x, y)$$

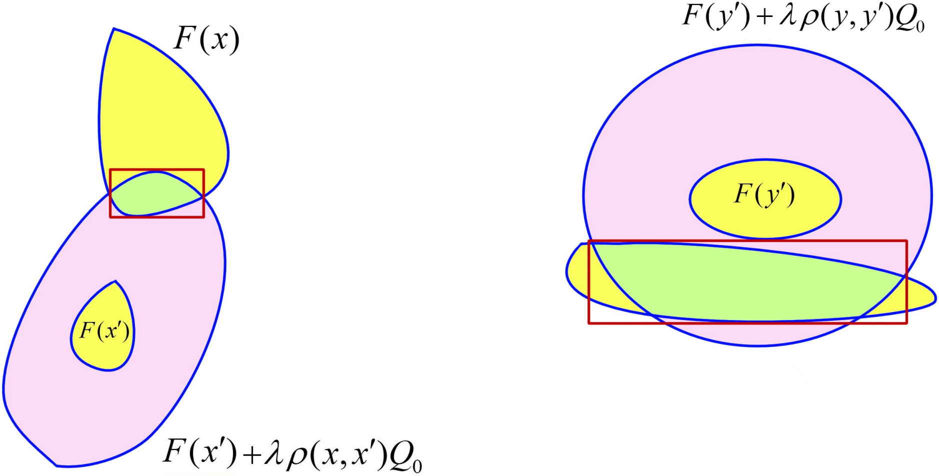
holds.

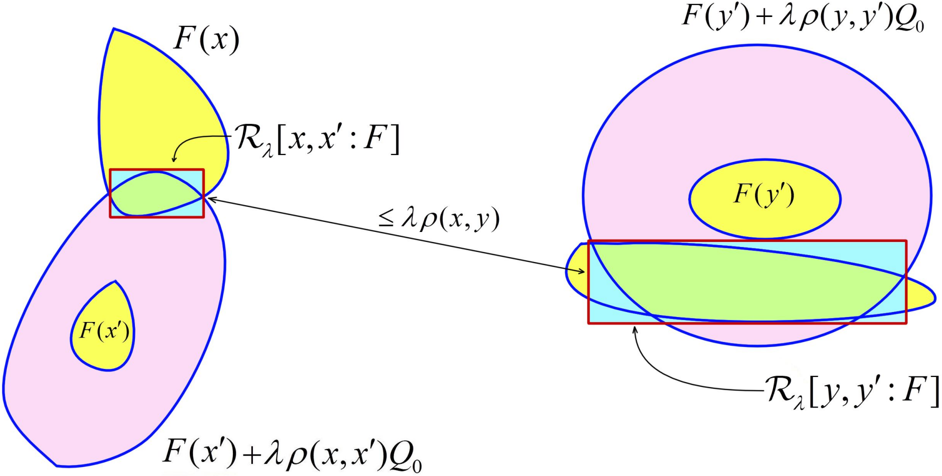
Furthermore,

$$\inf\{\|f\|_{\mathrm{Lip}(\mathcal{M},\mathbb{R}^2)} : f \text{ is a selection of } F \text{ on } \mathcal{M}\} \sim \inf \lambda$$


$F(x)$
$F(x')$
$F(x')+\lambda\rho(x,x')Q_0$


$F(y')+\lambda\rho(y,y')Q_0$
$F(y')$
$F(y)$


$F(x)$
$F(x')$
$F(x')+\lambda\rho(x,x')Q_0$
$F(y')+\lambda\rho(y,y')Q_0$
$F(y')$


$F(x)$
$\mathcal{R}_\lambda[x,x':F]$
$F(x')$
$F(x')+\lambda\rho(x,x')Q_0$
$\leq \lambda\rho(x,y)$
$F(y')+\lambda\rho(y,y')Q_0$
$F(y')$
$\mathcal{R}_\lambda[y,y':F]$

This criterion follows from a proof of the Finiteness Principle for Lipschitz selections for $Y = \mathbb{R}^2$ given below.

Given a set-valued mapping $F : \mathcal{M} \to \mathcal{K}_2(\mathbb{R}^2)$, we assume that the restriction $F|_{\mathcal{M}'}$ of F to every $\mathcal{M}' \subset \mathcal{M}$ with $\#\mathcal{M} \le 4$ has a Lipschitz selection $f_{\mathcal{M}'} : \mathcal{M}' \to \mathbb{R}^2$ with $\|f_{\mathcal{M}'}\|_{\mathrm{Lip}(\mathcal{M}',\mathbb{R}^2)} \le 1$.

Prove that F has a Lipschitz selection $f : \mathcal{M} \to \mathbb{R}^2$ with $\|f\|_{\mathrm{Lip}(\mathcal{M},\mathbb{R}^2)} \le 8$.

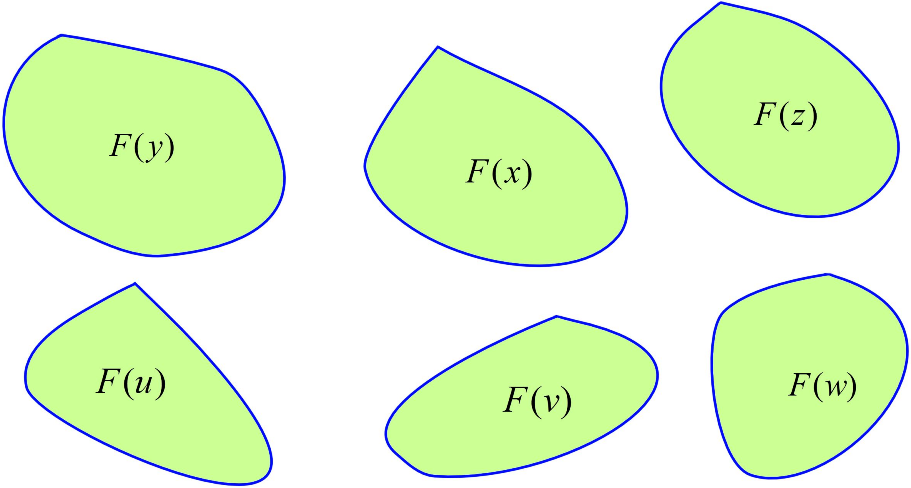
A Sketch of the Proof.

STEP 1. We construct the 1-balanced refinement of the mapping F:

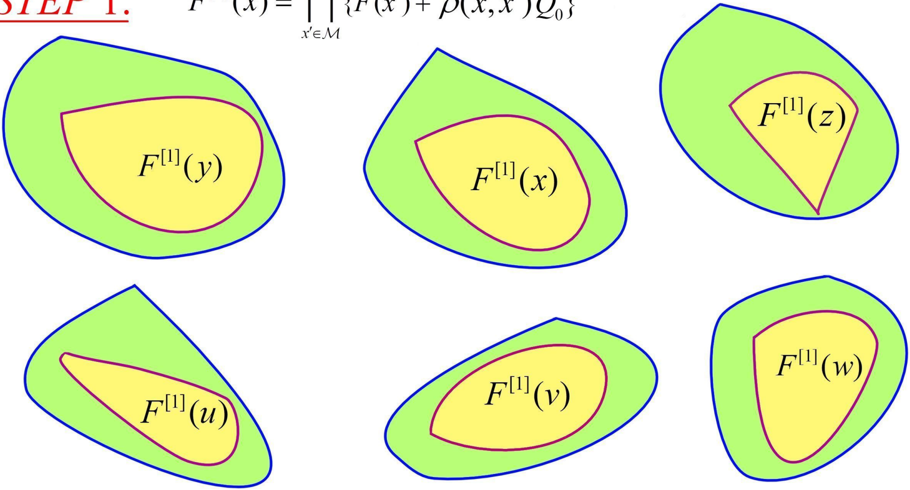
$$F^{[1]}(x) = \bigcap_{y \in \mathcal{M}} \left[F(y) + \rho(x,y)\,B\right], \qquad x \in \mathcal{M}.$$

STEP 2. We define a set-valued mapping $\mathcal{T}_F : \mathcal{M} \to \mathcal{R}(\mathbb{R}^2)$ which to every $x \in \mathcal{M}$ assigns the smallest box containing $F^{[1]}(x)$:

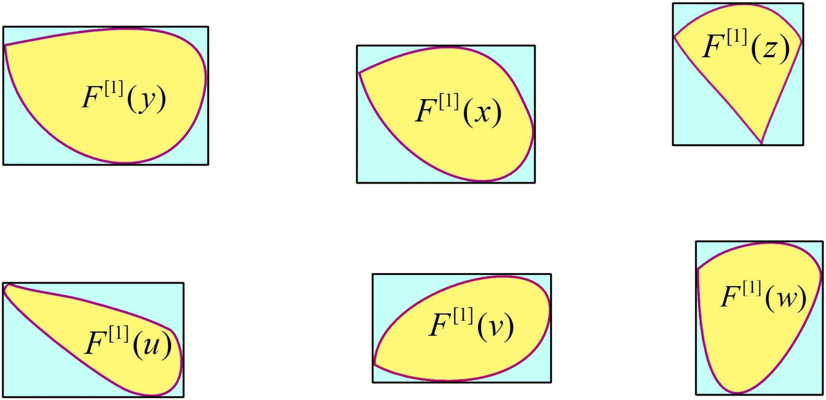
$$\mathcal{T}_F(x) = H\left[F^{[1]}(x)\right], \qquad x \in \mathcal{M}.$$


F(y)
F(x)
F(z)
F(u)
F(v)
F(w)

STEP 1. $F^{[1]}(x) = \bigcap_{x' \in \mathcal{M}} \{F(x') + \rho(x,x')Q_0\}$



STEP 2. $\mathcal{T}_F(x) = H[F^{[1]}(x)]$ - the smallest box containing $F^{[1]}(x)$



STEP 3. We prove that our assumption (i.e., the existence of a Lipschitz selection on every 4-point subset of $\mathcal{M}$ with Lipschitz constant ≤ 1) implies the following:

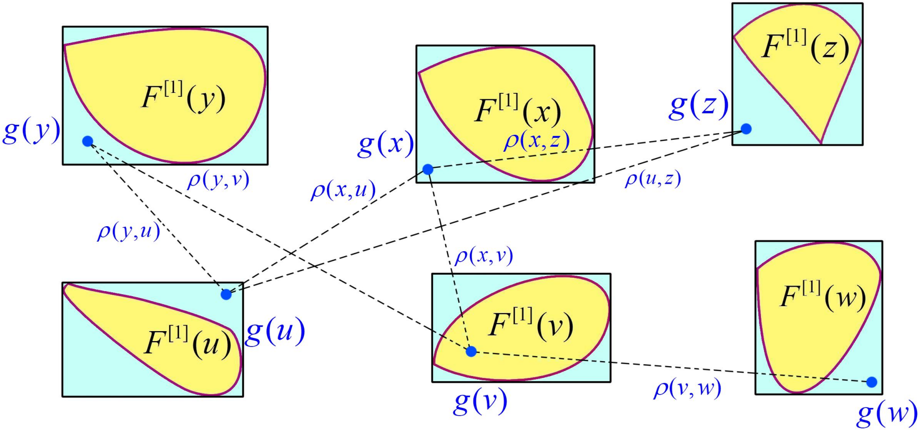
The restriction $\mathcal{T}_F|_{\mathcal{M}'}$ of the set-valued mapping $\mathcal{T}_F$ to every two point subset $\mathcal{M}' \subset \mathcal{M}$ has a Lipschitz selection $g_{\mathcal{M}'} : \mathcal{M}' \to \mathbb{R}^2$ with $\|g_{\mathcal{M}'}\|_{\mathrm{Lip}(\mathcal{M}',\mathbb{R}^2)} \le 1 \quad \Longleftrightarrow$

$$\mathrm{dist}(\mathcal{T}_F(x), \mathcal{T}_F(y)) \le \rho(x,y) \quad \text{for every} \quad x, y \in \mathcal{M}.$$

Hence we conclude that there exists a

Lipschitz selection $\; g : \mathcal{M} \to \mathbb{R}^2 \;$ of the mapping $\; \mathcal{T}_F : \mathcal{M} \to \mathcal{R}(\mathbb{R}^2)$

with $\|g\|_{\mathrm{Lip}(\mathcal{M},\mathbb{R}^2)} \le 1$.



$F^{[1]}(y)$
$g(y)$
$F^{[1]}(x)$
$g(x)$
$F^{[1]}(z)$
$g(z)$
$\rho(x,z)$
$\rho(y,v)$
$\rho(x,u)$
$\rho(u,z)$
$\rho(y,u)$
$\rho(x,v)$
$F^{[1]}(u)$
$g(u)$
$F^{[1]}(v)$
$g(v)$
$F^{[1]}(w)$
$\rho(v,w)$
$g(w)$

STEP 4. Given a convex closed set $G \subset \mathbb{R}^2$ we let $\mathrm{Pr}(\cdot : G)$ denote the metric projection operator (in ℓ^2_∞) onto G.

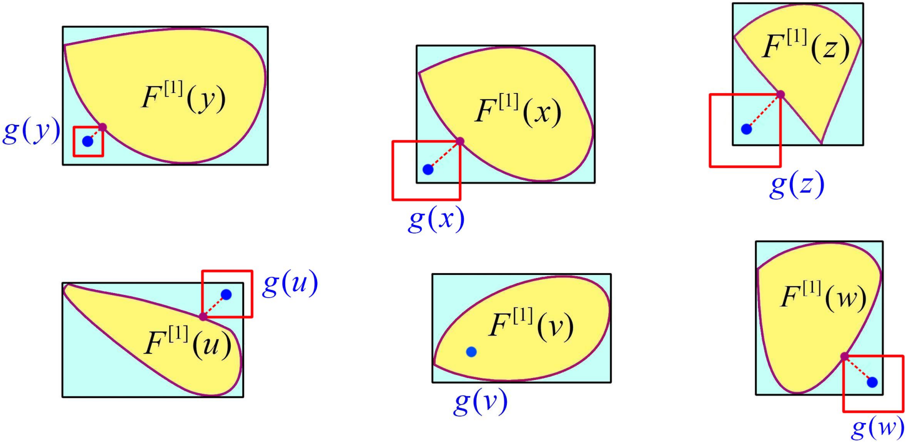
Finally, we define the required Lipschitz selection $f : \mathcal{M} \to \mathbb{R}^2$ by letting

$$f(x) = \mathrm{Pr}\left(g(x) : F^{[1]}(x)\right), \qquad x \in \mathcal{M}.$$

We prove that f is well defined on $\mathcal{M}$. We also show that

$$\|f(x) - f(y)\| \le 8\,\rho(x, y)$$

for every $x, y \in \mathcal{M}$ completing the proof of the theorem. $\square$


$F^{[1]}(y)$
$g(y)$
$F^{[1]}(x)$
$g(x)$
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$g(u)$
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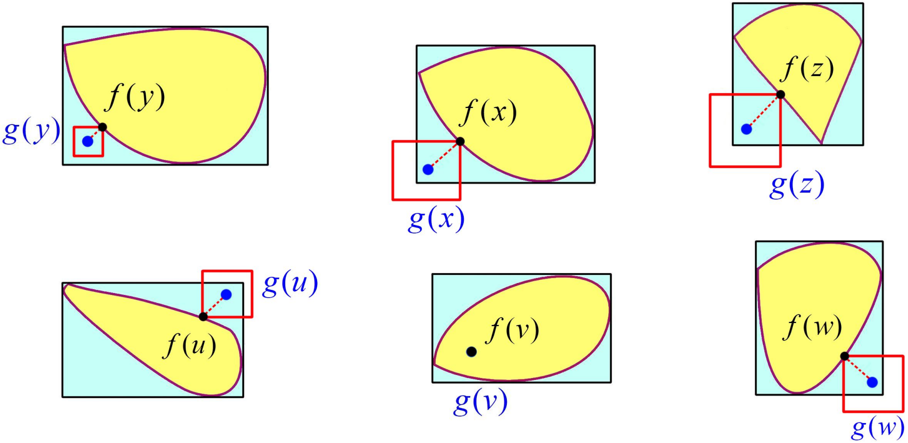
$F^{[1]}(v)$
$g(v)$
$F^{[1]}(w)$
$g(w)$



f(y)
g(y)
f(x)
g(x)
f(z)
g(z)
g(u)
f(u)
f(v)
g(v)
f(w)
g(w)

9. An Algorithm for Lipschitz Selections in $\mathbb{R}^2$

Let $(\mathcal{M}, \rho)$ be a pseudometric space, and let $F : \mathcal{M} \to \mathcal{K}_2(\mathbb{R}^2)$ be a set-valued mapping satisfying the following condition:

There exists a constant $\alpha > 0$ such that for every subset $\mathcal{M}' \subset \mathcal{M}$ with $\#\mathcal{M}' \le 4$ the restriction $F|_{\mathcal{M}'}$ has a Lipschitz selection $f_{\mathcal{M}'} : \mathcal{M}' \to \mathbb{R}^2$ with the Lipschitz seminorm

$$\|f_S\|_{\mathrm{Lip}(\mathcal{M}',\mathbb{R}^2)} \le \alpha.$$

STEP 1. We construct an α-balanced refinement of F:

$$F^{[1]}(x) = \bigcap_{y \in \mathcal{M}} \left[F(y) + \alpha\,\rho(x,y)\,Q_0\right], \qquad x \in \mathcal{M}.$$

STEP 2. We construct a set-valued mapping $\mathcal{T}_F : \mathcal{M} \to \mathcal{R}(\mathbb{R}^2)$ which to every $x \in \mathcal{M}$ assigns the smallest box containing $F^{[1]}(x)$:

$$\mathcal{T}_F(x) = H\left[F^{[1]}(x)\right], \quad x \in \mathcal{M}.$$

STEP 3. We construct an α-balanced refinement of $\mathcal{T}_F$:

$$\mathcal{T}_F^{[1]}(x) = \bigcap_{y \in \mathcal{M}} \left[\mathcal{T}_F(y) + \alpha\,\rho(x,y)\,Q_0\right], \quad x \in \mathcal{M}.$$

STEP 4. We construct a mapping $g : \mathcal{M} \to \mathbb{R}^2$ defined by

$$g(x) = \text{center}\left(\mathcal{T}_F^{[1]}(x)\right), \quad x \in \mathcal{M}.$$

STEP 5. We define a Lipschitz selection $f : \mathcal{M} \to \mathbb{R}^2$ of F by

$$f(x) = \Pr\left(g(x) : F^{[1]}(x)\right), \quad x \in \mathcal{M}.$$

Thank you!